Fast and Reliable Evaluation of the Distribution of Quadratic Forms of Gaussian Random Variables

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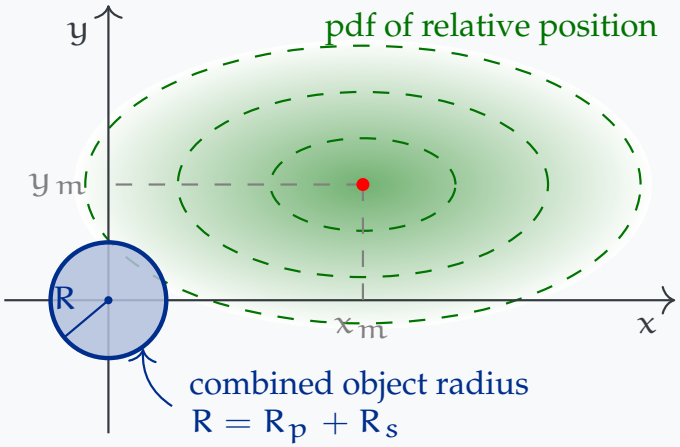
Example 1: Probability of Collision Between Space Objects



Short-term encounter model [Chan-2008]

- satellite & debris $\approx$ spheres
- high relative velocity:
  rectilinear relative motion
- Gaussian uncertainty on positions

$\rightsquigarrow$ 2D integral over a disk
in the encounter plane

$\rightsquigarrow$ maneuver if $\mathcal{P}_c > \delta$, e.g. $\delta = 10^{-4}$

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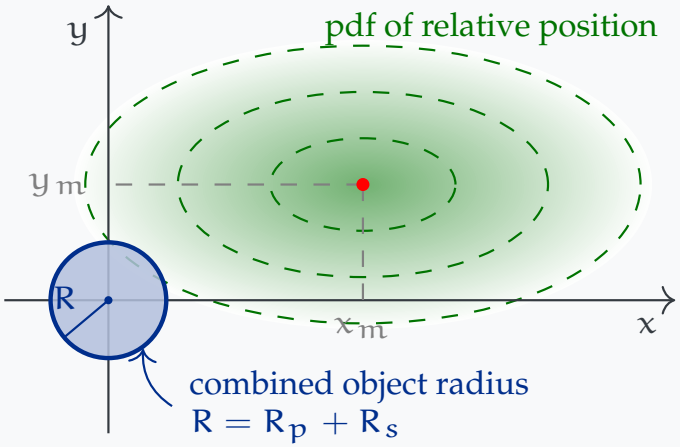

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Probability of Collision

$$\mathcal{P}_c = \frac{1}{2\pi\sigma_x\sigma_y} \iint_{x^2+y^2\leqslant R^2} \exp\left(-\frac{(x-x_m)^2}{2\sigma_x^2} - \frac{(y-y_m)^2}{2\sigma_y^2}\right) dx\,dy = \mathbb{P}\left[X^2+Y^2 \leqslant R^2\right]$$

$X \sim \mathcal{N}(x_m, \sigma_x^2)$, $Y \sim \mathcal{N}(y_m, \sigma_y^2)$ $\rightsquigarrow$ quadratic form with $d = 2$, $\xi = R^2$

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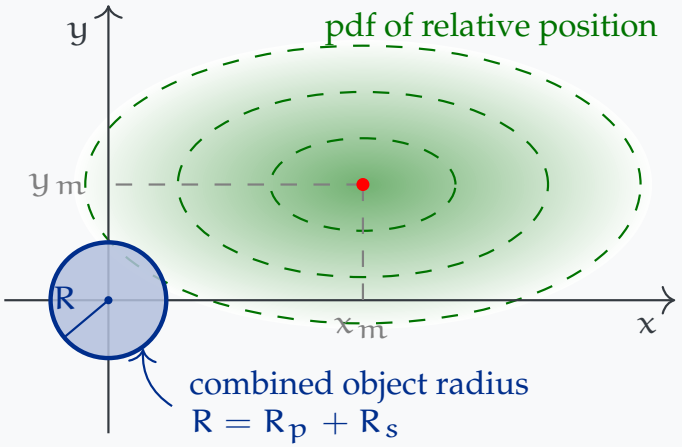

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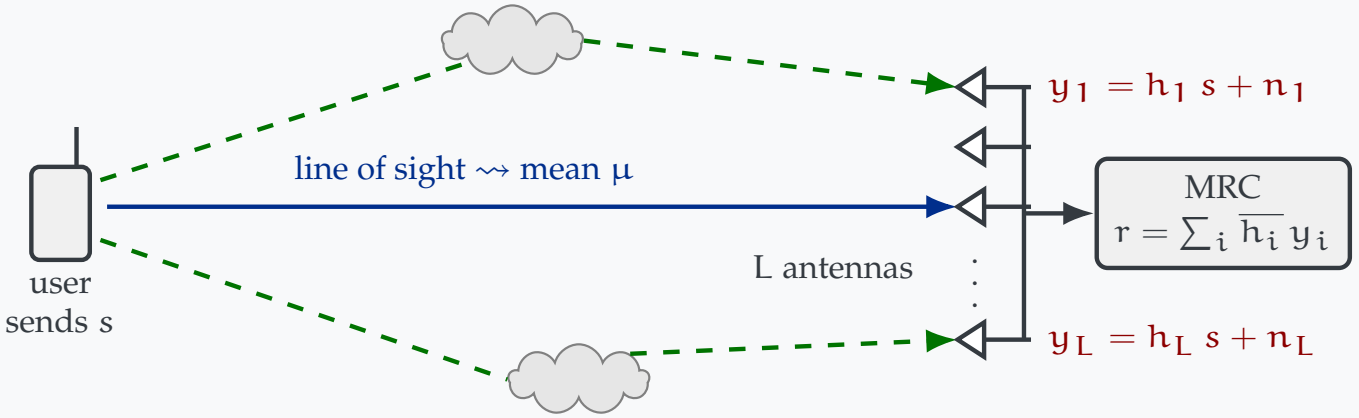
Holonomic approach [Serra-Arzelier-Joldes-et-al-2016]:

$$\mathcal{P}_c = e^{-\frac{R^2}{2\sigma_y^2}} \sum_{k\geqslant 1} \beta_k R^{2k}, \qquad \beta_k > 0 \text{ P-recursive } + \text{ explicit truncation bounds}$$

Example 2: Outage Probability in Massive MIMO



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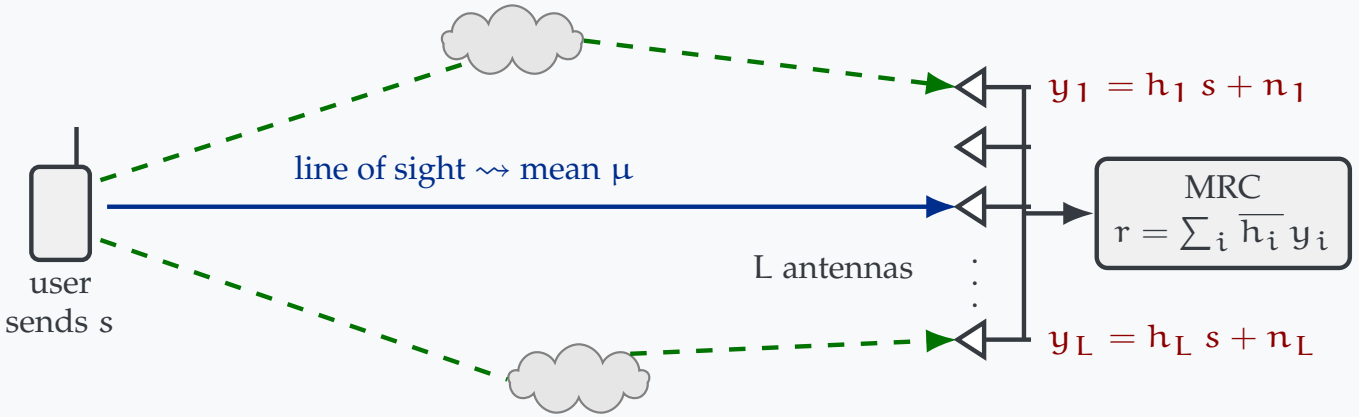

Maximum-Ratio Combining: Align Phases and Weight by Channel Strength

$$r = \sum_{i=1}^{L} \overline{h_i}\, y_i = \underbrace{\|h\|^2 s}_{\text{phases aligned}} + \sum_{i=1}^{L} \overline{h_i}\, n_i \quad \Longrightarrow \quad \mathsf{SNR} = \frac{\|h\|^4 P}{\|h\|^2 N_0} = \overline{\gamma}\, \|h\|^2$$

$\mathbb{E}|s|^2 = P$, $n_i \sim \mathcal{N}_{\mathbb{C}}(0, N_0)$ i.i.d., $\overline{\gamma} = P/N_0$

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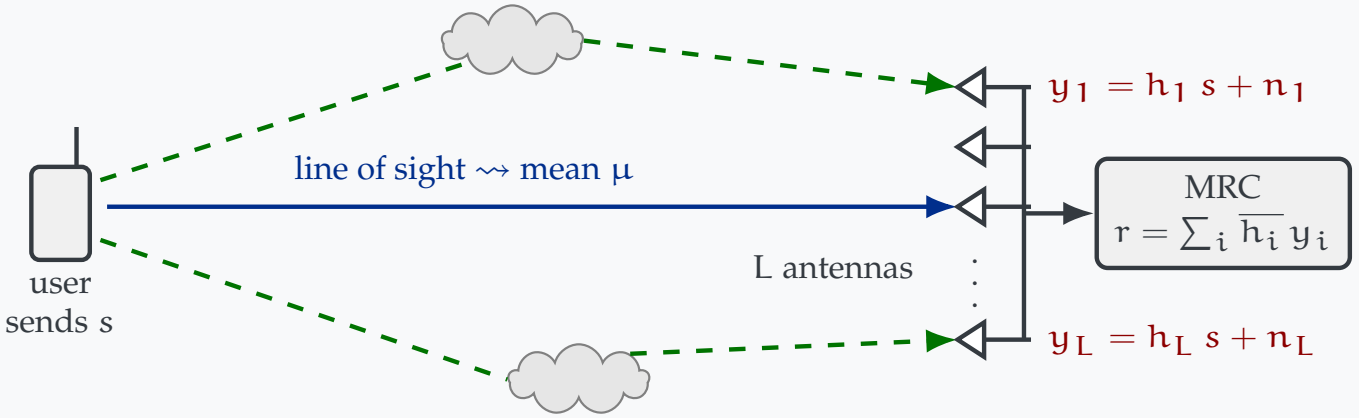

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Outage Probability as the CDF of a Gaussian Quadratic Form

$$P_{\text{out}} = \mathbb{P}\big[\mathsf{SNR} \leqslant \gamma_{\text{th}}\big] = \mathbb{P}\Big[\sum_{i=1}^{L} |h_i|^2 \leqslant \xi\Big] = \mathbb{P}\Big[\sum_{k=1}^{2L} X_k^2 \leqslant \xi\Big], \qquad \xi = \frac{\gamma_{\text{th}}}{\overline{\gamma}}$$

$h \sim \mathcal{N}_{\mathbb{C}}(\mu, R)$: diagonalize R, split real and imaginary parts $\Rightarrow X_k \sim \mathcal{N}(\mu_k, \sigma_k^2)$ independent

$\Rightarrow P_{\text{out}} = g(\xi)$ with $d = 2L$ (massive MIMO: $L \geqslant 100$), small values: lower tail

Finite-Precision CDF evaluation for Gaussian Quadratic Forms

Cumulative Distribution Function (CDF) for Noncentral Quadratic Sums

$$Q = \sum_{i=1}^{d} X_i^2 \qquad X_i \sim \mathcal{N}(\mu_i, \sigma_i^2) \text{ independent} \qquad g(\xi) = \mathbb{P}[Q \leqslant \xi]$$

$$g(\xi) = \frac{1}{\prod_{i=1}^{d} \sqrt{2\pi}\sigma_i} \int_{\sum_{i=1}^{d} x_i^2 \leqslant \xi} \exp\left(-\sum_{i=1}^{d} \frac{(x_i - \mu_i)^2}{2\sigma_i^2} \right) dx_1 \cdots dx_d$$

Up to renormalization, assume $\xi = 1$

Parameters: $m_i = \frac{\mu_i^2}{2\sigma_i^2}$, $\quad p_i = \frac{1}{2\sigma_i^2}$, $\quad p_1 \geqslant p_2 \geqslant \cdots \geqslant p_d$

Applications to: test statistics, spectral analysis and optimal control, space collision risk evaluation, radioastronomie, quantitative finance, SKAT (Sequence Kernel Association Test) in bioinformatics, ...

State of the Art and Contributions

Existing methods:

- Numerical inversion of the characteristic function, e.g. [Imhof-1961, Davies-1973]
- Divergent series using saddlepoint methods, e.g. [Daniels-1954, Kuonen-1999]
- Convergent power series expansions, e.g. [Ruben-1962, Johnson-Kotz-1970]
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- case $\mathrm{d} = 2$ applied to orbital collision probability in the short-term encounter model, with explicit truncation bounds
- Recurrence unrolling rounding error analysis using Mezzarobba's method
- Extension to the case $\mathrm{d} = 3$ (instantaneous probability of collision)

Roadmap of the approach and new contributions:



closed-form Laplace transform $\Rightarrow$ D-finite power series $\Rightarrow$ truncation and roundoff error control

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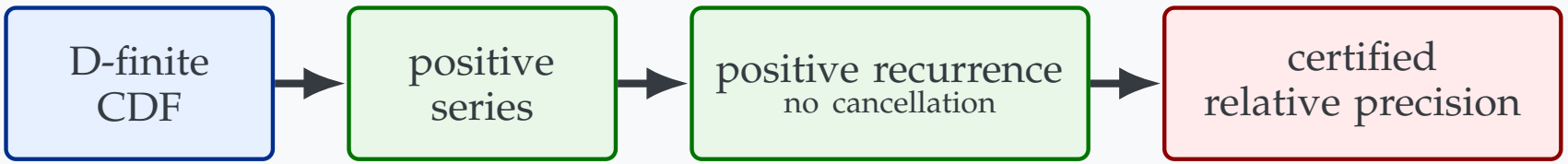

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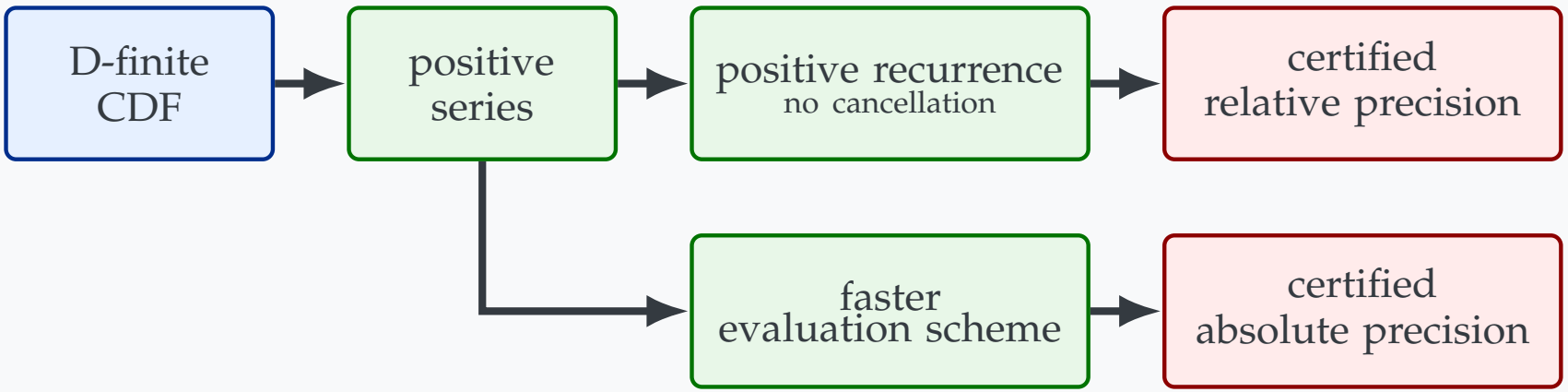

Algorithm CdfRec:

Extending the Recurrence to Arbitrary Dimension

Recurrence Through Closed-Form Laplace Transform – One Dimension

$$Q = X^2, \quad X \sim \mathcal{N}(\mu, \sigma^2) \quad p = \frac{1}{2\sigma^2}, \quad m = \frac{\mu^2}{2\sigma^2}$$

$$\begin{aligned} g(\xi) &= \mathbb{P}[Q \leqslant \xi] \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\sqrt{\xi}}^{\sqrt{\xi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x \end{aligned}$$

P-recursive $(g_n)_{n \geqslant 0}$

$$g(\xi) = \sum_{n \geqslant 0} g_n \xi^{n+1/2}$$

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Laplace Transform $\mathcal{L}_g(\lambda) = \int_0^{+\infty} e^{-\lambda\xi} g(\xi) d\xi$

$$\widehat{g}(\lambda) = \frac{\mathcal{L}_g(\lambda^{-1})}{\lambda^{3/2}} = \frac{\sqrt{p}\, \exp\left(-\frac{m}{1+p\lambda}\right)}{\sqrt{1+p\lambda}}$$

D-finite first-order ODE for $\widehat{g}$
explicit initial conditions

$$\widehat{g}(\lambda) = \sum_{n \geqslant 0} \Gamma\left(n + \frac{3}{2}\right) g_n \lambda^n$$

termwise inverse

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Differential Equation to Recurrence

$$\frac{\widehat{g}'(\lambda)}{\widehat{g}(\lambda)} = \frac{2mp - p - p^2\lambda}{2(1+p\lambda)^2} \quad \rightsquigarrow \quad \widehat{g}(\lambda) = \sum_{n\geqslant 0} \widehat{g}_n \lambda^n, \quad \widehat{g}'(\lambda) = \sum_{n\geqslant 0} n\widehat{g}_n \lambda^{n-1}$$

$$\sum_{n\geqslant 1} \left[2(n+1)\widehat{g}_{n+1} + p(4n+1-2m)\widehat{g}_n + p^2(2n-1)\widehat{g}_{n-1}\right] \lambda^n + (\dots) = 0$$

Recurrence Through Closed-Form Laplace Transform – One Dimension

$$Q = X^2, \quad X \sim \mathcal{N}(\mu, \sigma^2) \quad p = \frac{1}{2\sigma^2}, \quad m = \frac{\mu^2}{2\sigma^2}$$

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numerical cancellation

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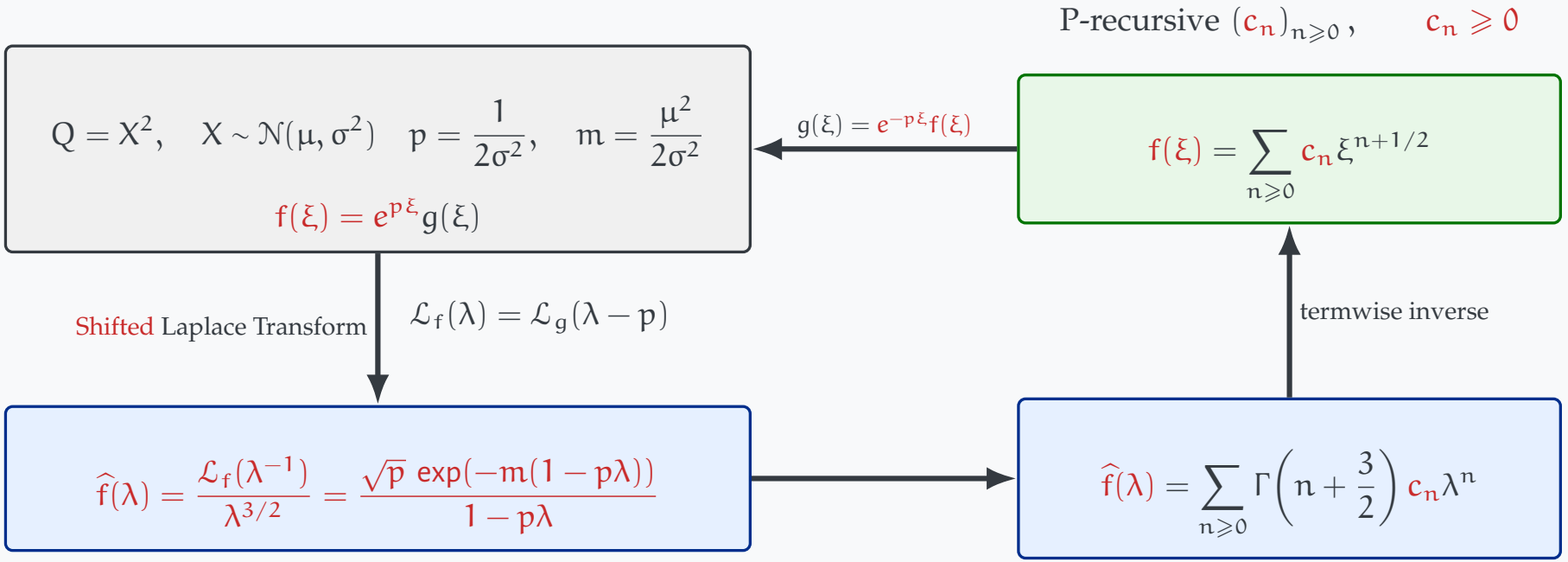
Recurrence Through Closed-Form Laplace Transform – Preconditioning



D-finite first-order ODE for $\widehat{f}$
explicit initial conditions

Differential Equation

$$\widehat{f}'(\lambda) = (pm + \underbrace{\frac{p}{1-p\lambda}}_{\gg 0})\widehat{f}(\lambda), \qquad \widehat{f}(0) = \sqrt{p}\, e^{-m}$$

Recurrence Through Closed-Form Laplace Transform – d Dimensions

$$Q = \sum_{i=1}^{d} X_i^2, \quad X_i \sim \mathcal{N}(\mu_i, \sigma_i^2), \quad p_i = \frac{1}{2\sigma_i^2}, \quad m_i = \frac{\mu_i^2}{2\sigma_i^2}$$

$$f(\xi) = e^{p_1 \xi} g(\xi), \qquad p_1 \geqslant p_2 \geqslant \dots p_d, \quad \alpha_i = p_1 - p_i$$

Shifted Laplace Transform

$$\widehat{f}(\lambda) = \frac{\mathcal{L}_f(\lambda^{-1})}{\lambda^{d/2+1}} = \frac{\prod_{i=1}^{d} \sqrt{p_i} \exp\left[-\sum_{i=1}^{d} m_i \frac{1-p_1\lambda}{1-\alpha_i\lambda}\right]}{(1-p_1\lambda)\prod_{i=2}^{d} \sqrt{1-\alpha_i\lambda}}$$

D-finite first-order ODE for $\widehat{f}$
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$g(\xi) = e^{-p_1 \xi} f(\xi)$

P-recursive $(c_n)_{n\geqslant 0}$, $\quad c_n \geqslant 0$

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$$\frac{\widehat{f}'(\lambda)}{\widehat{f}(\lambda)} = p_1 m_1 + \frac{p_1}{1-p_1\lambda} + \sum_{i=2}^{d} \left(\frac{\alpha_i}{2(1-\alpha_i\lambda)} + \frac{p_i m_i}{(1-\alpha_i\lambda)^2}\right) = \frac{P(\lambda)}{Q(\lambda)},$$

$$\widehat{f}(0) = \prod_{i=1}^{d} \sqrt{p_i}\, e^{-m_i}$$

CdfRec: Evaluation of Positive Series Using Scalar Recurrence

Require: d, $(p_i)_{1\leqslant i\leqslant d}$, $(m_i)_{1\leqslant i\leqslant d}$, number of terms N.

Ensure: $s_N \approx g(1)$.

1: $\alpha_i = p_1 - p_i, \quad 1 \leqslant i \leqslant d;$

2: $\varphi(\lambda) = p_1 m_1 + \dfrac{p_1}{1-p_1\lambda} + \displaystyle\sum_{i=2}^{d}\left(\frac{\alpha_i/2}{1-\alpha_i\lambda} + \frac{p_i m_i}{(1-\alpha_i\lambda)^2}\right);$

3: $P(\lambda) \leftarrow \mathrm{numerator}(\varphi), \quad Q(\lambda) \leftarrow \mathrm{denominator}(\varphi);$

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5: **for** $n = 1$ to $N-1$ **do**

6: $\quad c_n \leftarrow 0; \quad \mathrm{fac} \leftarrow 1;$

7: $\quad$ **for** $i = 1$ to $\min(2d, n)$ **do**

8: $\quad\quad \mathrm{fac} \leftarrow \left(\frac{d}{2} + n + 1 - i\right)\mathrm{fac};$

9: $\quad\quad c_n \leftarrow c_n + (-1)^{i-1}\dfrac{P_{i-1} + Q_i(n-i)}{\mathrm{fac}}\, c_{n-i};$

10: $\quad$ **end for**

11: $\quad c_n \leftarrow c_n/n;$

12: $\quad s \leftarrow s + c_n;$

13: **end for**

14: **return** $s_N = \exp(-p_1)s.$

construct rational ODE initial coefficient unroll recurrence

positive summation preconditioning

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8: $\quad\quad \mathrm{fac} \leftarrow \left(\frac{d}{2} + n + 1 - i\right)\mathrm{fac};$

9: $\quad\quad c_n \leftarrow c_n + (-1)^{i-1}\dfrac{P_{i-1} + Q_i(n-i)}{\mathrm{fac}}\, c_{n-i};$

10: $\quad$ **end for**

11: $\quad c_n \leftarrow c_n/n;$

12: $\quad s \leftarrow s + c_n;$

13: **end for**

14: **return** $s_N = \exp(-p_1)s.$

choose N from truncation bound

roundoff error bound

mixed signs in the recurrence!!

construct rational ODE

initial coefficient

unroll recurrence

positive summation

preconditioning

Algorithm CdfPosRec:

A Positive Recurrence for Reducing the Roundoff Error

Coupled Recurrence With Positive Coefficients

$$\widehat{f'} = \left(p_1 m_1 + \frac{p_1}{1 - p_1 \lambda} + \sum_{i=2}^{d} \left(\frac{\alpha_i / 2}{1 - \alpha_i \lambda} + \frac{p_i m_i}{(1 - \alpha_i \lambda)^2} \right) \right) \widehat{f}$$

$$\gamma_n := \Gamma(n + \tfrac{d}{2} + 1)$$

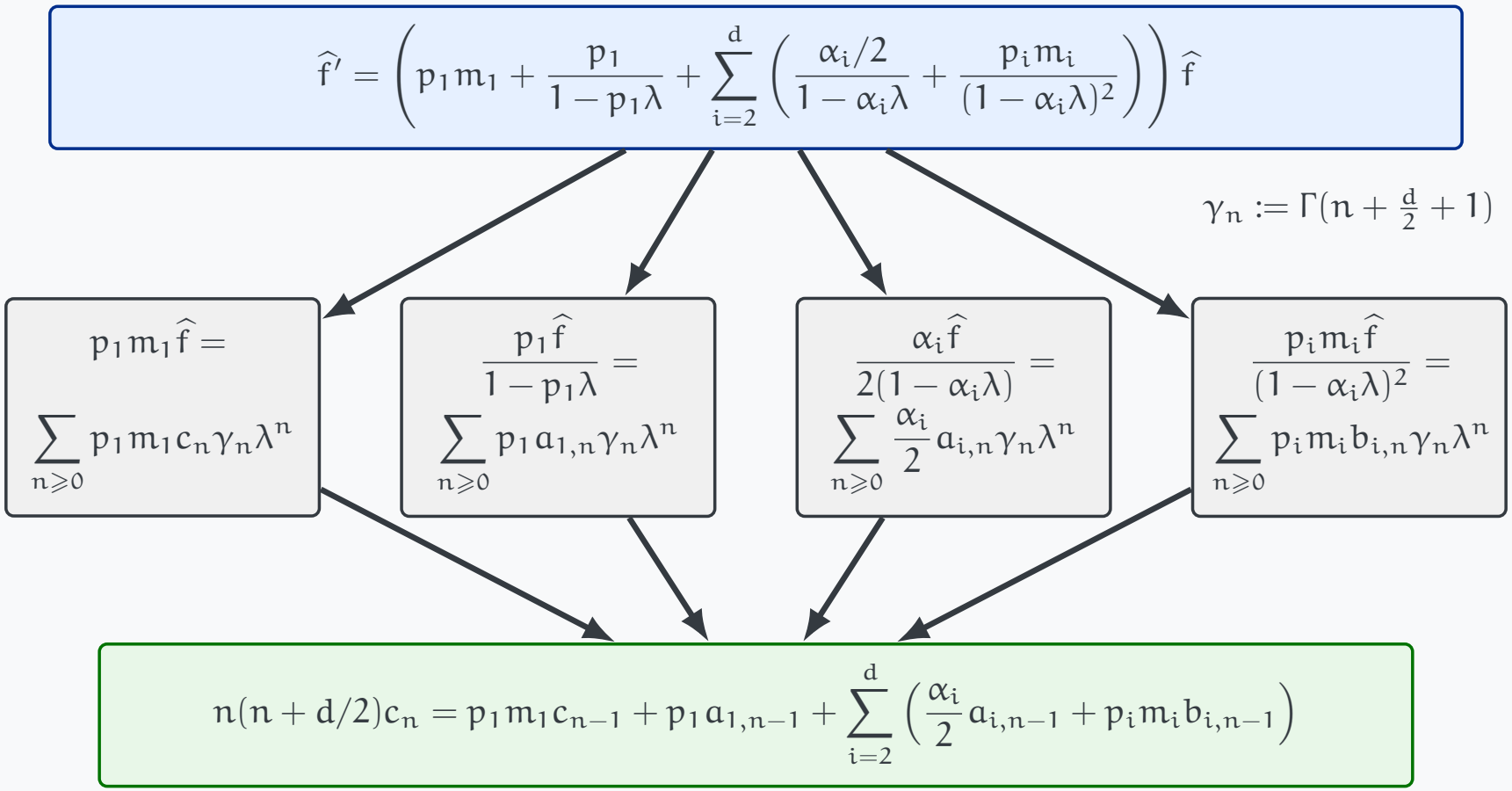
$$p_1 m_1 \widehat{f} = \sum_{n \geqslant 0} p_1 m_1 c_n \gamma_n \lambda^n$$

$$\frac{p_1 \widehat{f}}{1 - p_1 \lambda} = \sum_{n \geqslant 0} p_1 a_{1,n} \gamma_n \lambda^n$$

$$\frac{\alpha_i \widehat{f}}{2(1 - \alpha_i \lambda)} = \sum_{n \geqslant 0} \frac{\alpha_i}{2} a_{i,n} \gamma_n \lambda^n$$

$$\frac{p_i m_i \widehat{f}}{(1 - \alpha_i \lambda)^2} = \sum_{n \geqslant 0} p_i m_i b_{i,n} \gamma_n \lambda^n$$

Coupled Recurrence With Positive Coefficients



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$$\widehat{f}' = \left(p_1 m_1 + \frac{p_1}{1 - p_1 \lambda} + \sum_{i=2}^{d} \left(\frac{\alpha_i / 2}{1 - \alpha_i \lambda} + \frac{p_i m_i}{(1 - \alpha_i \lambda)^2} \right) \right) \widehat{f}$$

$$\gamma_n := \Gamma(n + \tfrac{d}{2} + 1)$$

$$p_1 m_1 \widehat{f} = \sum_{n \geqslant 0} p_1 m_1 c_n \gamma_n \lambda^n$$

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$$n(n + d/2) c_n = p_1 m_1 c_{n-1} + p_1 a_{1,n-1} + \sum_{i=2}^{d} \left(\frac{\alpha_i}{2} a_{i,n-1} + p_i m_i b_{i,n-1} \right)$$

$$a_{1,n} = \frac{p_1}{n + d/2} a_{1,n-1} + c_n$$

$$a_{i,n} = \frac{\alpha_i}{n + d/2} a_{i,n-1} + c_n$$

$$b_{i,n} = \frac{\alpha_i}{n + d/2} b_{i,n-1} + a_{i,n}$$

CdfPosRec: Evaluation Using Coupled Recurrence With Positive Coefficients

Require: d, $(p_i)_{1\leqslant i\leqslant d}$, $(m_i)_{1\leqslant i\leqslant d}$, number of terms $\mathbf{N}$.

Ensure: $s_N \approx g(1)$.

1: $\alpha_i = p_1 - p_i, \quad 1 \leqslant i \leqslant d;$

2: $c_0 = \dfrac{\left(\prod_{i=1}^{d} \sqrt{p_i}\right) \exp\left(-\sum_{i=1}^{d} m_i\right)}{\Gamma\left(\frac{d}{2}+1\right)};$ $\qquad a_{i,0} \leftarrow c_0, \quad b_{i,0} \leftarrow c_0;$

3: $s \leftarrow c_0;$

4: **for** $n = 1$ to $\mathbf{N} - 1$ **do**

5: $\quad c_n \leftarrow \dfrac{p_1 m_1 c_{n-1} + p_1 a_{1,n-1} + \sum_{i=2}^{d} \left(\frac{\alpha_i}{2} a_{i,n-1} + p_i m_i b_{i,n-1}\right)}{n\left(n+\frac{d}{2}\right)};$

6: $\quad a_{1,n} \leftarrow \dfrac{p_1}{n+\frac{d}{2}} a_{1,n-1} + c_n;$

7: $\quad$ **for** $i = 2$ to d **do**

8: $\quad\quad a_{i,n} \leftarrow \dfrac{\alpha_i}{n+\frac{d}{2}} a_{i,n-1} + c_n; \qquad b_{i,n} \leftarrow \dfrac{\alpha_i}{n+\frac{d}{2}} b_{i,n-1} + a_{i,n};$

9: $\quad$ **end for**

10: $\quad s \leftarrow s + c_n;$

11: **end for**

12: **return** $s_N = \exp(-p_1)s.$

initial terms — positive recurrence unrolling

positive summation — preconditioning

Roundoff Error Analysis of CdfPosRec

Floating-Point Model [Higham-2002]

$\mathsf{RN}(x \circ y) = (x \circ y)(1+\delta)$
$|\delta| \leqslant u = 2^{-t}$ (precision t, no overflow)
$\exp$, $\sqrt{\cdot}$, $\Gamma(\frac{d}{2}+1)$ faithfully rounded

$$\left|\prod_{j=1}^{k}(1+\delta_j)^{\pm 1} - 1\right| \leqslant \frac{ku}{1-ku} \quad (\approx ku)$$

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Key Property: Nonnegative Terms

If all $s_j \geqslant 0$:

$$\sum_j s_j(1+\varepsilon_j) = \left(\sum_j s_j\right)(1+\varepsilon)$$
with $|\varepsilon| \leqslant \max_j |\varepsilon_j|$

(mixed signs: amplified by $\sum_j |s_j| / |\sum_j s_j|$)

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$$c_0 = \frac{\left(\prod_{i=1}^{d}\sqrt{p_i}\right)\exp\left(-\sum_{i=1}^{d} m_i\right)}{\Gamma\left(\frac{d}{2}+1\right)};\quad a_{i,0} \leftarrow c_0;\ b_{i,0} \leftarrow c_0;\ s \leftarrow c_0;$$

for $n = 1$ to $\mathbf{N-1}$ **do**

$$c_n \leftarrow \frac{p_1 m_1 c_{n-1} + p_1 a_{1,n-1} + \sum_{i=2}^{d}\left(\frac{\alpha_i}{2} a_{i,n-1} + p_i m_i b_{i,n-1}\right)}{n\left(n+\frac{d}{2}\right)};$$

$$a_{1,n} \leftarrow \frac{p_1}{n+\frac{d}{2}} a_{1,n-1} + c_n;$$

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end for

$s \leftarrow \sum_{n=0}^{N-1} c_n$;

return $s_N = \exp(-p_1)s$.

relative errors $\widetilde{c}_n = c_n(1+\varepsilon_n)$

$$\varepsilon_0 \lesssim (d+7+(d-1)M)\,u$$

$M = \sum_i m_i$ rounded inside exp

$$\varepsilon_n \lesssim \varepsilon_{n-1} + (2d+4)\,u$$

$$\lesssim \varepsilon_n + u$$

$$\lesssim \varepsilon_n + u, \quad \lesssim \varepsilon_n + 2u$$

$$\lesssim \max_n \varepsilon_n + N\,u$$

$+\,3u$ (exp, $\times$)

Roundoff Error Analysis of CdfPosRec

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$\lesssim \varepsilon_n + u$

$\lesssim \varepsilon_n + u, \quad \lesssim \varepsilon_n + 2u$

$\lesssim \max_n \varepsilon_n + N\,u$
$+ 3u$ (exp, $\times$)

Relative error on $s_N \lesssim \big(\underbrace{(2d+4)N}_{\text{recurrence}} + \underbrace{(d-1)M + d + 11}_{\text{init. \& final}}\big)\,u$

$\Rightarrow$ precision $t = \tau + \log_2\big(d\,(N+M)\big) + O(1)$ bits for rel. accuracy $2^{-\tau}$

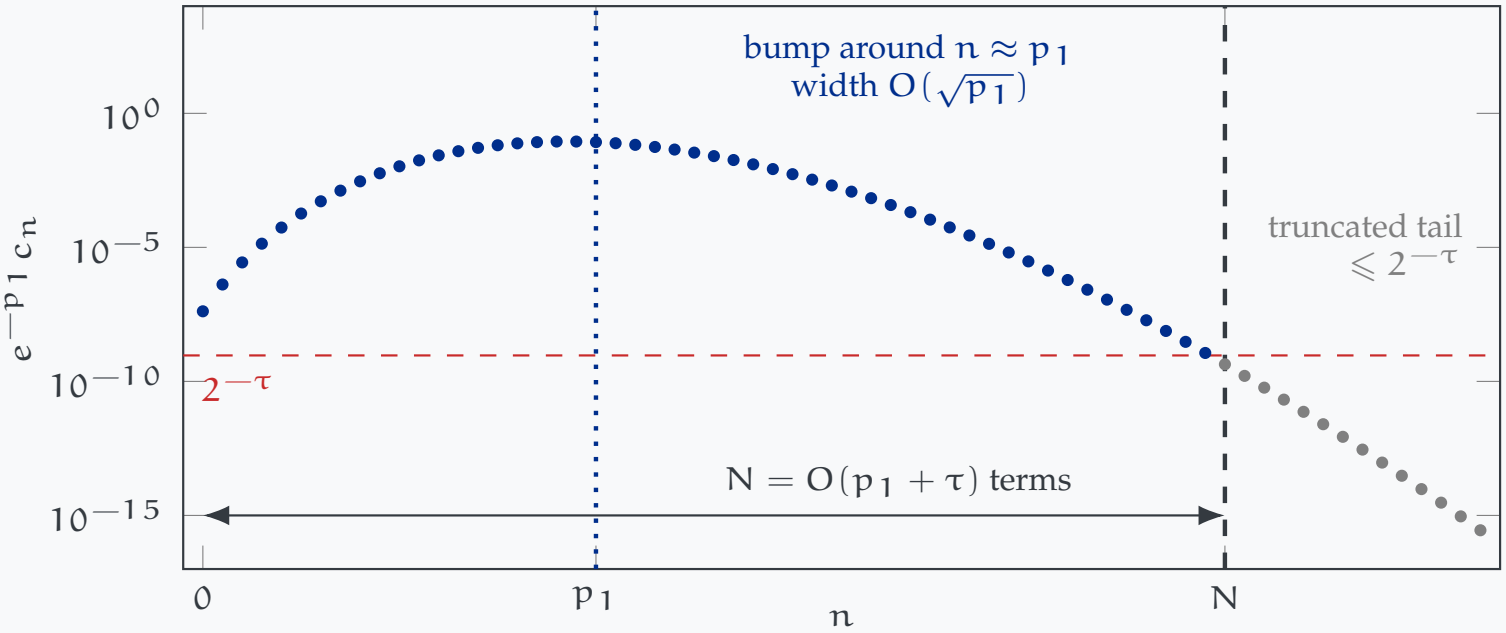
Complexity of CdfPosRec with Target Relative Accuracy

Truncation Error Bounds [Arzelier-Bréhard-Joldes-2026]

Let $r_N := \left| g(1) - \exp(-p_1) \sum_{n=0}^{N-1} c_n \right|$ be the truncation error bound:

(i) *(absolute error)* $r_N \leqslant 2^{-\tau}$ for $N = O\left(p_1 + \tau\right)$

(ii) *(relative error)* $r_N \leqslant 2^{-\tau} g(1)$ for $N = O\left(p_1 + \sum_i m_i + \sum_i \ln\left(\frac{p_1}{p_i}\right) + \tau\right)$



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Complexity of Algorithm CdfPosRec [Arzelier-Bréhard-Joldes-2026]

CdfPosRec computes $g(1)$ with relative accuracy $2^{-\tau}$ in

$$\widetilde{O}\Bigg(d\Big(\underbrace{p_1 + \textstyle\sum_i m_i + \sum_i \ln\left(\frac{p_1}{p_i}\right) + \tau}_{N \text{ terms}}\Big) \underbrace{\tau}_{\substack{\text{working precision } t \text{ up to} \\ \text{log factors in } p_1,\, d,\, m_i,\, \tau}} \Bigg) \quad \text{bit operations}$$

Numerical Experiment: CdfRec vs. CdfPosRec

- Comparing `cdf_rec` and `cdf_pos_rec`, implemented in Julia (here in `binary64`)
- `iso` = central isotropic ($\mu_i = 0$, σ_i equal), `aniso` = central anisotropic ($\mu_i = 0$), `noncent` = noncentral anisotropic, `hard` = large N, `das` = examples from [Das-2005]

Case	Type	d	p_{ref}	N	T_{posrec} (s)	T_{rec} (s)	Dig. $_{posrec}$	Dig. $_{rec}$
S1	iso	10	0.3555	30	1.01e-5	5.78e-5	15.9	15.9
S2	iso	30	0.4908	50	3.41e-5	1.71e-4	15.9	15.5
S3	aniso	10	0.9974	70	2.38e-5	1.61e-4	14.9	10.3
S4	aniso	10	0.8746	100	3.33e-5	2.38e-4	14.5	4.5
S5	aniso	10	0.5960	150	4.99e-5	3.67e-4	14.2	0.0
S6	aniso	10	0.2553	330	1.18e-4	8.68e-4	15.8	0.0
S7	noncent	10	0.5799	150	5.29e-5	3.85e-4	14.7	0.0
S8	noncent	10	0.5288	150	5.33e-5	3.82e-4	14.8	0.1
S9	noncent	10	0.4352	150	5.21e-5	3.85e-4	14.0	0.4
S10	noncent	10	0.2957	150	5.34e-5	3.84e-4	14.1	1.0
S11	aniso	30	0.9606	200	1.46e-4	1.36e-3	14.0	0.0
S12	aniso	30	0.5599	340	2.49e-4	2.45e-3	14.2	0.0
S13	noncent	30	0.5068	340	2.50e-4	2.44e-3	13.7	0.0
S14	noncent	30	0.3383	340	2.48e-4	2.49e-3	14.6	0.0
S15	hard	10	4.451e-04	**7,910**	2.93e-3	2.30e-2	12.8	0.0
S16	hard	30	2.162e-13	**7,900**	6.03e-3	7.44e-2	12.8	0.0
S17	hard	10	4.101e-06	**183,500**	9.94e-2	6.21e-1	10.7	0.0
S18	hard	30	4.515e-23	**183,500**	1.96e-1	1.80e+0	10.7	0.0
D1	das	3	0.4936	30	9.04e-6	2.90e-5	15.9	15.0
D5	das	8	0.5913	60	1.99e-5	1.18e-4	15.2	11.4
D9	das	10	0.5848	80	2.87e-5	1.93e-4	14.4	10.0
D11	das	34	0.5736	210	1.79e-4	1.56e-3	15.5	0.0

Algorithm CdfFast:

An Alternative Approach for Taming High Dimensions

Overview of Algorithm CdfFast – Taming High Dimensions

$$\widehat{f}(\lambda) = \frac{\prod_i \sqrt{p_i} e^{-m_i}}{1 - p_1\lambda} \exp\left[\underbrace{p_1 m_1 \lambda + \sum_{i=2}^{d} \left(\frac{p_i m_i \lambda}{1 - \alpha_i \lambda} - \frac{1}{2} \ln(1 - \alpha_i \lambda) \right)}_{\psi(\lambda)\text{: analytic over } \overline{B}(0, \frac{1}{p_1})}\right]$$

1. Compute $\psi(\lambda)$ up to degree N in $\widetilde{O}(\max(N, d))$ FP operations

   $\rightsquigarrow$ fast transposed multipoint evaluation to compute sums $\displaystyle\sum_{i=1}^{d} \frac{w_i}{1 - \alpha_i \lambda} \bmod \lambda^N$

2. Compute $\exp(\psi(\lambda))$ up to degree N in $\widetilde{O}(N)$ FP operations

   $\rightsquigarrow$ use a FFT-based exponentiation scheme over a disk of radius $\dfrac{1 - \varepsilon}{p_1}$

3. Compute $\widehat{f}(\lambda)$ and eventually $g(1)$ in $\widetilde{O}(N)$ FP operations

A Fast Numerical Transposed Multipoint Evaluation Scheme

Transposed Multipoint Evaluation Problem

$$\sum_{i=1}^{d} \frac{w_i}{1-\alpha_i z} \,\%\, z^N = \sum_{n=0}^{N-1} \Bigg(\underbrace{\sum_{i=1}^{d} w_i \alpha_i^n}_{u_n} \Bigg) z^n$$

Vandermonde matrix V_α: $\quad u = V_\alpha^T \cdot w \quad \leftrightsquigarrow \quad y = V_\alpha \cdot p$

Multipoint Evaluation Problem

$$p(z) = \sum_{n=0}^{N-1} p_n z^n \quad \rightsquigarrow \quad y_i = p(\alpha_i) = \sum_{n=0}^{N-1} p_n \alpha_i^n, \quad 1 \leqslant i \leqslant d$$

Tellegen's transposition principle

For a linear map $A : \mathbb{K}^n \to \mathbb{K}^m$, if $A \cdot v$ can be computed by a SLP of size L, then $A^T \cdot w$ can be computed by a SLP of size $L - n + m$.

Moroz' Fast Numerical Multipoint Evaluation Algorithm (assume $N = d$)

Multipoint Evaluation Problem

$$p(z) = \sum_{n=0}^{d-1} p_n z^n \qquad \rightsquigarrow \qquad y_i = p(\alpha_i) = \sum_{n=0}^{d-1} p_n \alpha_i^n, \quad 1 \leqslant i \leqslant d$$

- Symbolic algorithms in $\widetilde{O}(d)$ arithmetic operations, e.g. [Bostan-Lecerf-Schost-2003]
- in FP arithmetic $\rightsquigarrow$ eval. error $= O(2^d)$ $\rightsquigarrow$ need extra d bits

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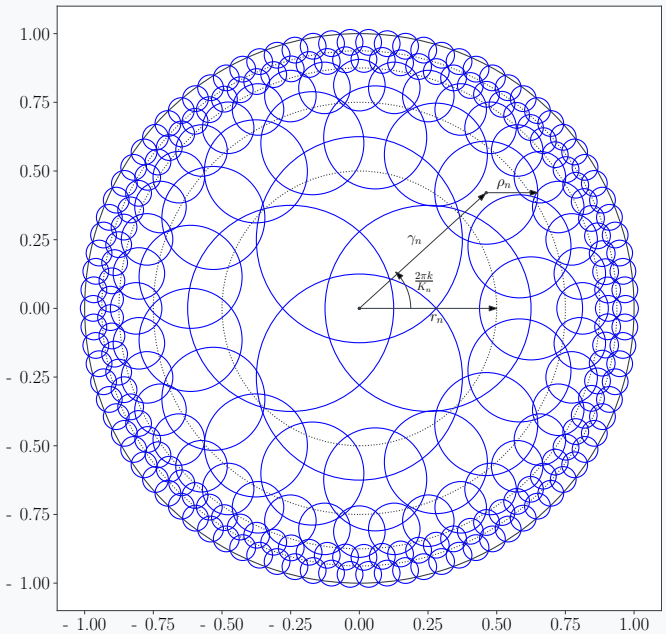

Credit: Guillaume Moroz

0. Compute the t-hyperbolic covering of the unit disk
   $\rightsquigarrow$ radii tend to zero close to the boundary

1. Shift and truncate p to degree t on each small disk

2. On each small disk, evaluate all $p(\alpha_i)$ for α_i in the disk (with error $O(2^t)$)

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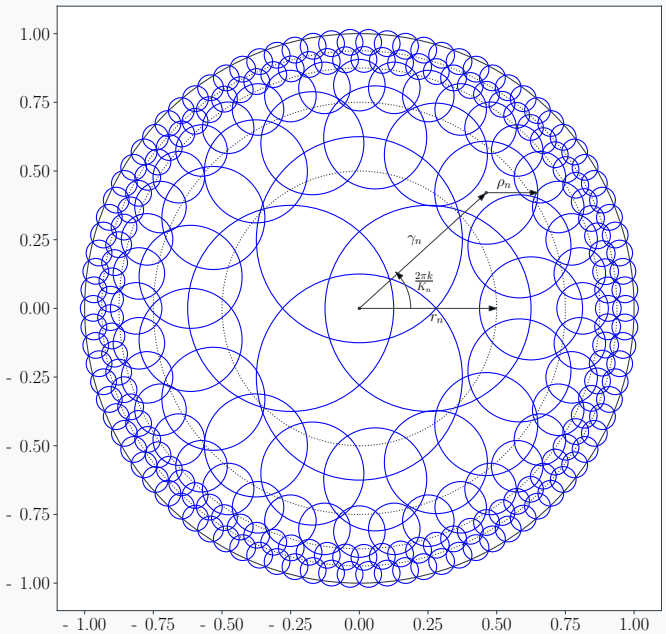

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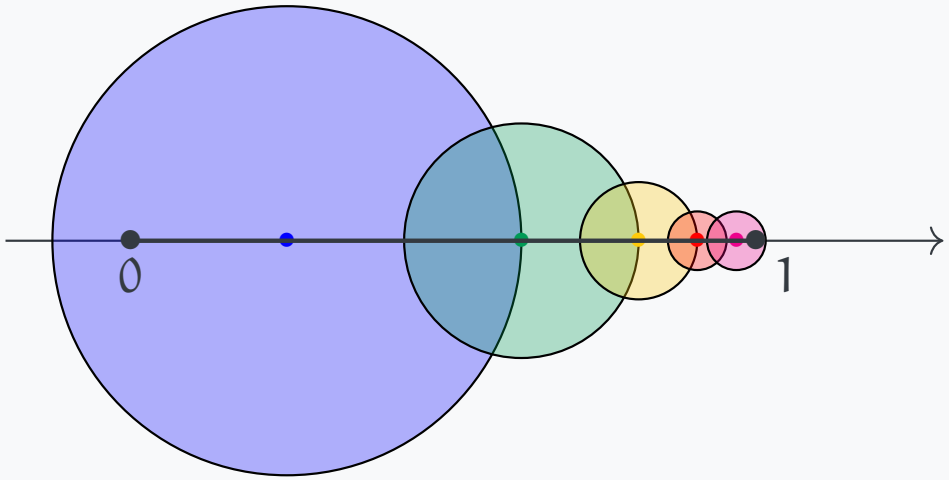
$$\text{Error} \leqslant \|p\|_1 2^{-t} \text{ in } \widetilde{O}\big(d(t + \ln \|p\|_1)\big) \text{ bit op.}$$

Transposing Numerical Fast Multipoint Evaluation

0. Compute the t-hyperbolic covering of $[0, 1]$

   $\rightsquigarrow$ Disks $D_k := \overline{B}(\gamma_k, \rho_k)$; $\quad \alpha_i \mapsto \beta_i = \frac{\alpha_i - \gamma_k}{\rho_k}$ with k such that $\alpha_i \in D_k$



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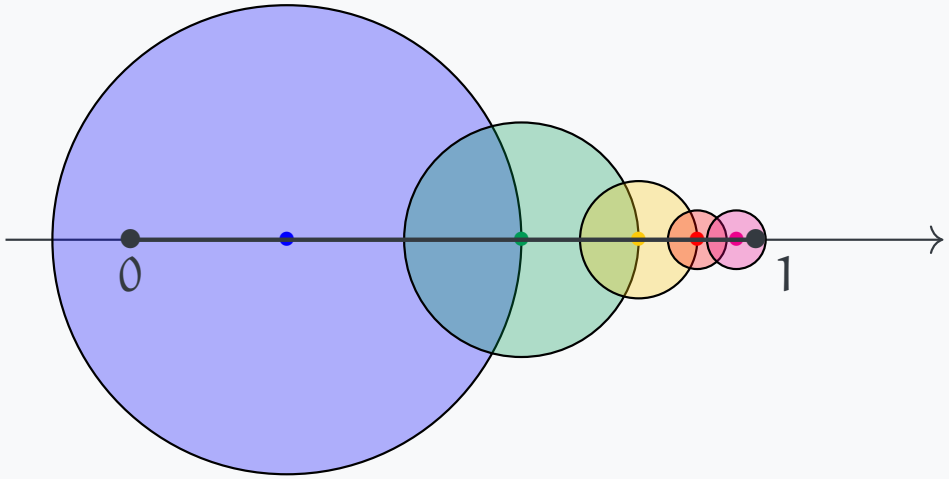

2^{T}. Compute $U_k := \sum_{i \mid \alpha_i \in D_k} \frac{w_i}{1 - \beta_i z} \bmod z^t$ using a transposed multipoint evaluation algorithm (with error $O(2^t)$)

1^{T}. (Shift and Truncate)^{T} = Pad and Transposed Shift

$$\mathsf{shift}^{\mathsf{T}}_{\gamma_k, \rho_k} : U(z) \mapsto \frac{1}{1 - \gamma_k z} U\left(\frac{\rho_k z}{1 - \gamma_k z}\right) \quad \Rightarrow \quad \frac{1}{1 - \beta_i z} \mapsto \frac{1}{1 - \alpha_i z}$$

$\Rightarrow$ Sum the $V_k := \mathsf{shift}^{\mathsf{T}}_{\gamma_k, \rho_k}\left(U_k\right)$

Transposing Numerical Fast Multipoint Evaluation

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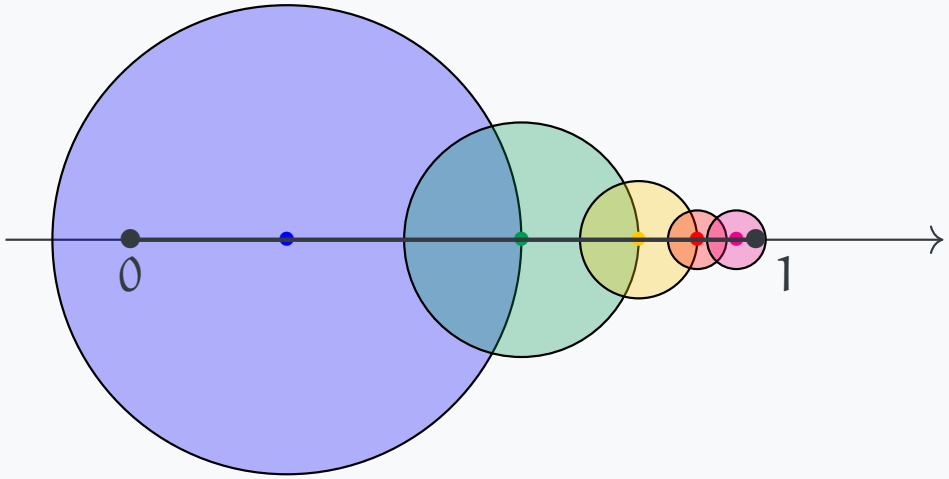

2^{T}. Compute $U_k := \sum_{i \mid \alpha_i \in D_k} \frac{w_i}{1-\beta_i z} \bmod z^t$ using a transposed multipoint evaluation algorithm (with error $O(2^t)$)

1^{T}. (Shift and Truncate)^{T} = Pad and Transposed Shift

$$\mathrm{shift}^{\mathsf{T}}_{\gamma_k,\rho_k} : U(z) \mapsto \frac{1}{1-\gamma_k z} U\left(\frac{\rho_k z}{1-\gamma_k z}\right) \quad \Rightarrow \quad \frac{1}{1-\beta_i z} \mapsto \frac{1}{1-\alpha_i z}$$

$\Rightarrow$ Sum the $V_k := \mathrm{shift}^{\mathsf{T}}_{\gamma_k,\rho_k}\left(U_k\right)$

Error $\leqslant \|w\|_1 2^{-t}$ in $\widetilde{O}\left(d(t + \ln \|w\|_1)\right)$ bit op.

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$$h(\lambda) = \exp\left(\psi(\lambda)\right) \qquad \psi(\lambda) \text{ analytic over } \overline{B}(0, p_1^{-1})$$

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  $\rightsquigarrow$ aliasing + rounding error: $p_1^{\frac{d}{2}} \left(\frac{\nu^N}{1-\nu} + CN \log N u\right) \quad [u = 2^{-t}]$

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trade-off when $t \ll N$: $\nu = 2^{-\frac{t}{N}} \approx 1 - \frac{t}{N} \log(2)$

$$\frac{1-\nu}{\nu} p_1 \leqslant \frac{t}{2} \ln(2) \quad \rightsquigarrow \quad |\widetilde{s}_N - s_N| = O\left(2^{-\frac{t}{2}}\right)$$

Complexity of CdfFast vs. CdfPosRec with Target Absolute Accuracy

Complexity of Algorithm CdfPosRec [Arzelier-Bréhard-Joldes-2026]

$\textsc{CdfPosRec}(d, (p_i), (m_i))$ computes $g(1)$ with absolute accuracy 2^{-t} in

$$\widetilde{O}\left(\underbrace{(p_1 + t)}_{N \text{ terms}} \, d \, \underbrace{t}_{\substack{\text{working prec. up to} \\ \text{log factors in } p_1, d, m_i, t}}\right) \quad \text{bit operations}$$

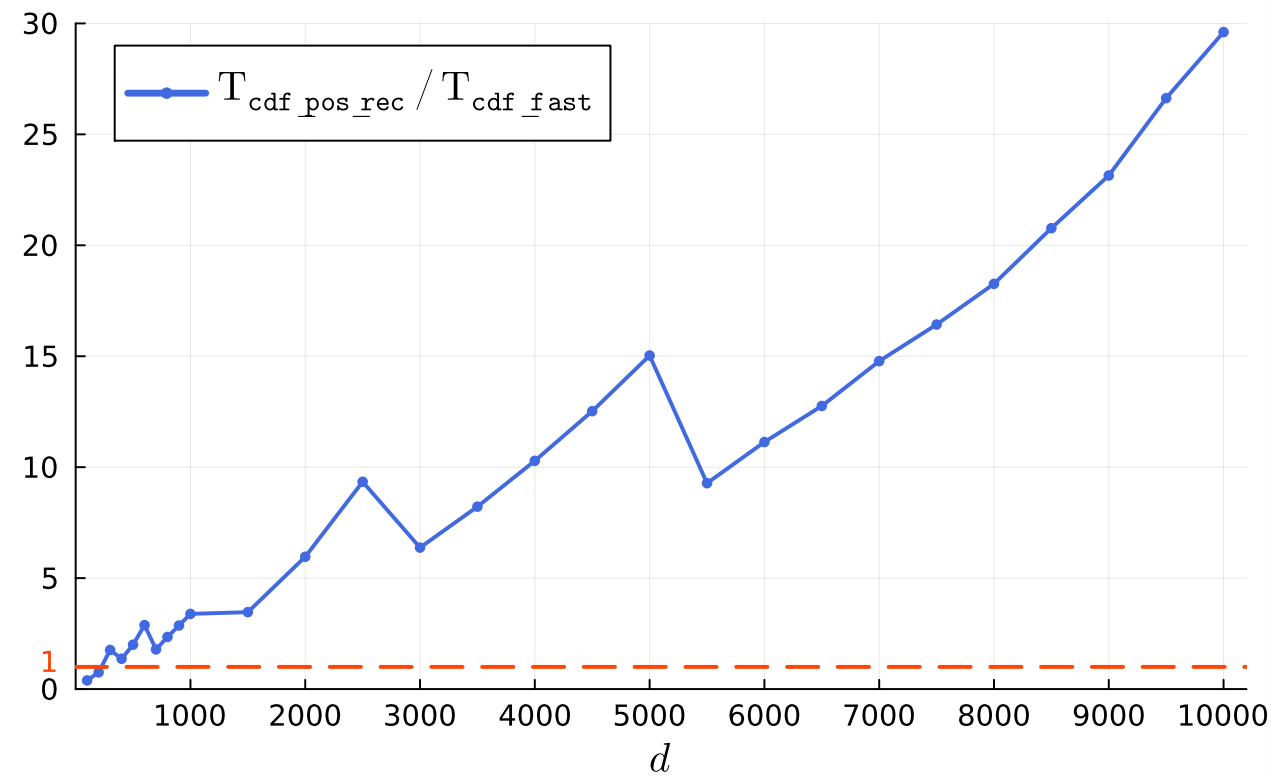
Complexity of Algorithm CdfFast [Arzelier-Bréhard-Joldes-2026]

$\textsc{CdfFast}(d, (p_i), (m_i))$ computes $g(1)$ with absolute accuracy 2^{-t} in

$$\widetilde{O}\left(\underbrace{(p_1 + t + d)}_{N \text{ terms}} \underbrace{t}_{\substack{\text{working prec. up to} \\ \text{log factors in } p_1, d, m_i, t}}\right) \quad \text{bit operations}$$

Numerical Experiment in High Dimensions: CdfFast vs. CdfPosRec

- $\sigma_i \sim \mathcal{U}([1,2])$; $\mu_i \sim \mathcal{N}(0,1)$; rescale by $\left(\sum_i (\sigma_i^2 + \mu_i^2)\right)^{-\frac{1}{2}}$ $\rightsquigarrow$ $g(1) \approx \frac{1}{2}$
- Set $N = 2d$ and use 64-bit precision $\rightsquigarrow$ absolute error bounded by 10^{-10}



- Speedup of CdfFast over CdfPosRec reaches 30 for $d = 10,000$ (3.5 s vs. 104 s)
- Fast polynomial multiplication (FFT) and fast transposed multipoint evaluation (for base case t-blocks) not implemented yet

Extension to Weighted Sums of Noncentral χ^2 Variables

Problem: Groups of Variables With Equal Variance

$$Q = \sum_{j=1}^{r} \sigma_j^2 \, \chi'^2_{k_j}(\delta_j) = \sum_{j=1}^{r} \sum_{l=1}^{k_j} X_{j,l}^2, \qquad X_{j,l} \sim \mathcal{N}(\mu_{j,l}, \sigma_j^2) \text{ indep.}$$

r variances, $d = \sum_j k_j$ variables; $p_j = \frac{1}{2\sigma_j^2}$, $m_j = \sum_l \frac{\mu_{j,l}^2}{2\sigma_j^2}$ (total noncentrality)

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Modified Laplace Transform: Multiplicities Become Exponents

$$\widehat{f}(\lambda) = \frac{\prod_{j=1}^{r} p_j^{k_j/2} \exp\left[-\sum_{j=1}^{r} m_j \frac{1-p_1\lambda}{1-\alpha_j\lambda}\right]}{(1-p_1\lambda)\prod_{j=2}^{r}(1-\alpha_j\lambda)^{k_j/2}}, \qquad \alpha_j = p_1 - p_j,$$

$$\frac{\widehat{f}'(\lambda)}{\widehat{f}(\lambda)} = p_1 m_1 + \frac{p_1}{1-p_1\lambda} + \sum_{j=2}^{r}\left(\frac{k_j\,\alpha_j/2}{1-\alpha_j\lambda} + \frac{p_j m_j}{(1-\alpha_j\lambda)^2}\right)$$

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What Changes in the Method – Ongoing Work

- **truncation bounds:** unchanged ($d = \sum_j k_j$, $\ln \frac{p_1}{p_j}$ counted k_j times)
- CDFREC: scalar recurrence of order $2r$ instead of $2d$
- CDFPOSREC: same recurrence on r groups ($\frac{\alpha_j}{2} \to \frac{k_j\alpha_j}{2}$) $\Rightarrow$ $O(Nr)$ instead of $O(Nd)$
- **roundoff:** $O(r)$ roundings per step $\Rightarrow \lesssim ((2r + O(1))N + (r-1)M)u$
- CDFFAST: multipoint evaluation on r points $\Rightarrow$ $\widetilde{O}(N + r)$

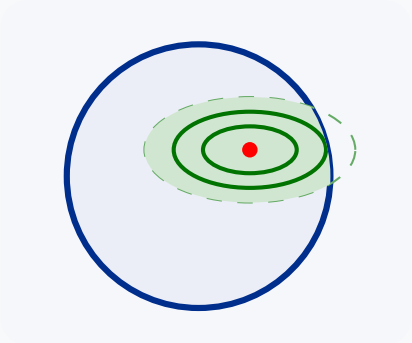
Conclusion and Future Work

Contributions:

- Generalization of [Serra-Arzelier-Joldes-et al-2016] to arbitrary dimension d
- Reduced evaluation error using a positive coupled recurrence
- Faster algorithm for high-dimensional cases
- Prototype implementations in Julia and Matlab

Future Research Directions:

- Extension to non-positive quadratic forms of Gaussian random variables
- Relative error bounds for CDFFAST?
- Connections with Fast Multipole Methods (FMM)?
- Investigation applications, especially in high dimension



`Quadratiques.jl`

fast and reliable CDF evaluation

`gitlab.laas.fr/arzelier/quadratic`

Scan for repository

Thank you!