Genericity in multivariate cryptography

(or how to appease Reviewer B by removing heuristics)

Simon Abelard[1,2], Pierre Pébereau[3], Mohab Safey El Din[2]
[1]EPITA, [2]LIP6, Sorbonne Université, [3]COSIC, KU Leuven.



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Unbalanced Oil and Vinegar [Patarin '97] [Kipnis, Patarin, Goubin '99]

Multivariate signature schemes: a template

- **Public key**: a system of polynomial equations of degree two $\longrightarrow$ **Hard to solve**
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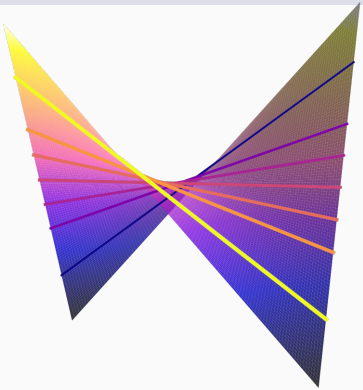

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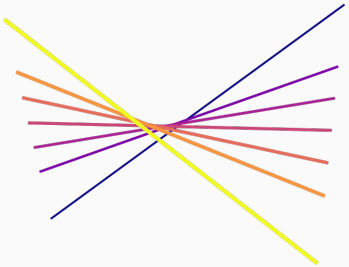
$$(1, 0, 0) + \mathbb{R}(0, 1, 1).$$



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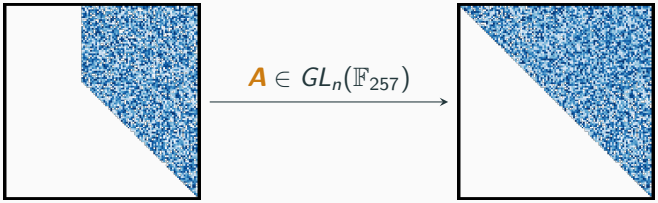
Private key: polynomials $f_1, \ldots, f_m \in \mathbb{F}_q[X_1, \ldots, X_n]$ **linear** in $X_1, \ldots, X_o$, $A \in GL_n(\mathbb{F}_q)$.

Public key: polynomials $p_1, \ldots, p_m \in \mathbb{F}_q[X_1, \ldots, X_n]$, with $p_i = X^T \cdot P_i \cdot X = f_i \circ A$.

Private polynomial
$F_1 \in (\mathbb{F}_{257})^{n \times n}$



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Equivalent formulation: there exists a linear subspace $\mathcal{O}$ of dim. o such that

$$p_1(\mathcal{O}) = \cdots = p_m(\mathcal{O}) = 0.$$

For UOV, require that $o \geq m$.

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Zariski topology on $\mathbb{K}^n$

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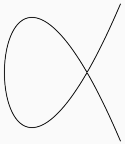
Example: generic smoothness

The curve of equation $Y^2 - X^3 + 3X - 2 = 0$ is **singular**.

Are all curves of equation $Y^2 - X^3 - aX - b = 0$ singular?



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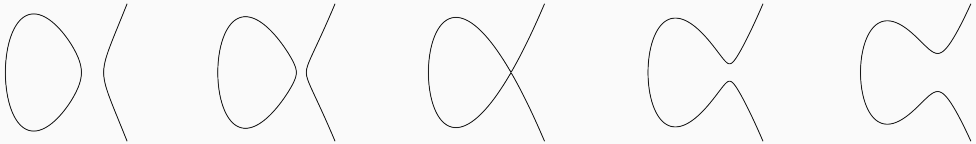

Figure 1: Curves in $\mathbb{R}^2$ defined by $Y^2 - X^3 - aX - 2$ for $a \in [-3.4, -2.6]$.

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Methodology

Consider a, b as **variables** and study the equation $\mathrm{Jac}_E(a, b) = (0, 0)$.

$$\mathrm{Jac}_E(a, b) = (0, 0) \in \mathbb{K}(X, Y)[a, b] \implies \delta(a, b) = 4a^3 + 27b^2 = 0.$$

$$\mathrm{Jac}_{\mathcal{F}} = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial X_1} & \cdots & \frac{\partial f_m}{\partial X_n} \end{bmatrix}$$

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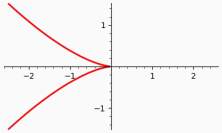
Therefore, if $\delta(a, b)$ is non-zero, $E_{a,b}$ is **smooth**.



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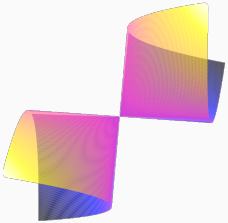

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Example: Quadric hypersurfaces in $\mathbb{R}^3$.

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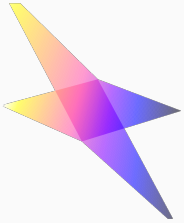

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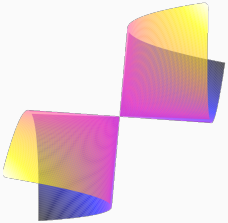

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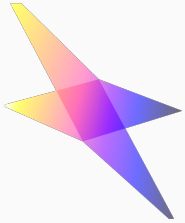

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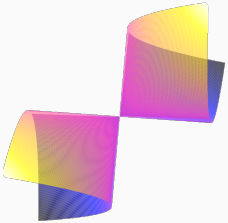

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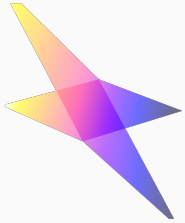
Thus, a **generic** quadric hypersurface in $\mathbb{R}^3$ contains lines but does not contain planes.



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$$1 \leq k \leq m,\ 1 \leq i \leq j \leq o, \quad P_k(X_i, X_j) = 0.$$

- The dimension of $\mathbb{G}_o(\mathbb{P}^n(\mathbb{K}))$ is $(o+1)(n-o)$, ie the number of coefficients in an echelon basis of an $(o+1)$-dimensional linear subspace of $\mathbb{K}^{n+1}$.

Contributions and Outline of the talk

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Common regularity assumptions are generic... This work

If $\mathcal{P} \in \mathcal{U}$, then $\mathcal{P}$ forms a regular sequence and $V(\mathcal{I})$ is an equidimensional complete intersection [...].

Support analysis of the direct attack and of some key recovery attacks. In particular, geometric attacks are well-defined.

Contributions and Outline of the talk

Singularities of generic UOV keys. [P. 2025]

For a generic UOV key $\mathcal{P} \in \mathcal{U}$, we have $\mathrm{Sing}(V(\mathcal{I})) \subset \mathcal{O}$.

This result unlocks subsequent genericity results.

A family of examples. This work

The previous result holds in any finite field.

Address a limit of [P. 2025] which worked only over large fields.

Common regularity assumptions are generic... This work

If $\mathcal{P} \in \mathcal{U}$, then $\mathcal{P}$ forms a regular sequence and $V(\mathcal{I})$ is an equidimensional complete intersection [...].

Support analysis of the direct attack and of some key recovery attacks. In particular, geometric attacks are well-defined.

...more exotic assumptions on MinRank also are. This work

For a generic UOV key $\mathcal{P} \in \mathcal{M}$, the determinantal ideals related to the Jacobian evaluated on $\mathcal{O}$ have the expected dimension.

Support the analysis of Rectangular MinRank attacks. No false positives in the Kipnis-Shamir attack.

Generic UOV keys

Indeterminate UOV secret key

Fix $O = \{1, \ldots, o\}$ (oil indices) and $V = \{o+1, \ldots, n\}$ (vinegar indices).
For $1 \leq k \leq m$,

$$\Phi^{(k)}(\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \underbrace{\sum_{i \in O,\ j \in V} \alpha_{i,j}^{(k)}\, x_i x_j}_{\text{oil} \times \text{vinegar}} + \underbrace{\sum_{i \leq j \in V} \beta_{i,j}^{(k)}\, x_i x_j}_{\text{vinegar} \times \text{vinegar}}.$$

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- The oil space is $\mathcal{O} = \{\boldsymbol{x} \mid x_{o+1} = \cdots = x_n = 0\}$.
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The properties we study are preserved by A through the chain rule, so we may study them for the secret map Φ_θ.

For generic UOV keys, all singularities lie in $\mathcal{O}$.

Structured equations yield a structured Jacobian. [P. 2025]

The Jacobian of $\mathcal{F}$ is structured when $\boldsymbol{x} \in \mathcal{O}$:

$$\mathrm{Jac}_{\mathcal{F}}(\boldsymbol{x}) = \begin{matrix} 1 \\ \vdots \\ m \end{matrix} \overset{\begin{matrix} 1 \cdots\cdots o & o+1 \cdots\cdots\cdots n \end{matrix}}{\begin{bmatrix} 0 & J_2 \end{bmatrix}}$$

- There are singularities in $\mathcal{O}$.
- Are there singular points outside of $\mathcal{O}$, that may hinder attacks or enable distinguishers?

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For a generic UOV key, we have $\mathrm{Sing}(V(\mathcal{I})) \subset \mathcal{O}$, if $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{F}_p, p \gg 1$.

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It suffices to give examples in smaller fields to extend the result by proving $\mathcal{U} \neq \emptyset$.

Goal: construct a family of UOV keys whose Jacobians have full rank outside of $\mathcal{O}$.

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A family of examples in finite fields of odd characteristic.

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Sketch of proof.

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Genericity of regularity properties.

Reminder: Definition of $\mathcal{U}$.

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Consider UOV parameters (n, m, o, q), with $m \leq o < n - m$.
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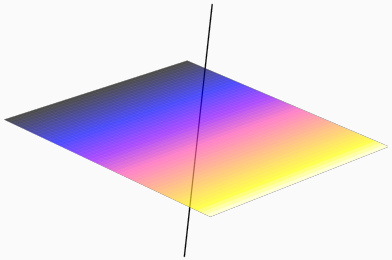

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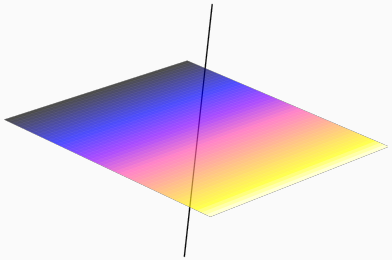

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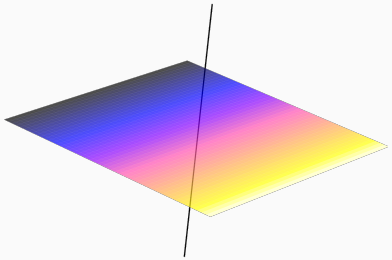

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Supporting heuristics in the algebraic analysis of attacks.

Direct attack against UOV

Given $\boldsymbol{t} = \mathcal{H}(M) \in \mathbb{F}_q^m$, find $\boldsymbol{x} \in \mathbb{F}_q^n$ such that

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$\longrightarrow$ Special case of Fröberg's conjecture [Fröberg 85].

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- Irregular UOV keys (ie $n - m < o$) are not regular sequences (early MAYO parameters, QR-UOV/SNOVA big field instances, VOX, ...).

Determinantal varieties of UOV and application to MinRank attacks.

Dimension of determinantal varieties

- If $M \in \mathbb{K}[X_1, \ldots, X_n]^{p \times q}$, the ideal of minors of degree $r+1$ of M has codimension at most $(p-r) \times (q-r)$. [Bruns, Vetter 1988]

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- Tangent spaces and singular points [P. 24, 25]

All these attacks solve MinRank problems on Jacobian matrices.

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All these attacks solve MinRank problems on Jacobian matrices.

Jacobian matrices are not generic: cannot apply [FSS11] out of the box.

Determinantal varieties of UOV and application to MinRank attacks.

Dimension of determinantal varieties

- If $M \in \mathbb{K}[X_1, \ldots, X_n]^{p \times q}$, the ideal of minors of degree $r+1$ of M has codimension at most $(p-r) \times (q-r)$. [Bruns, Vetter 1988]
- Equality for **generic** polynomial matrices. [Faugère, Safey el Din, Spaenlehauer 2011]

MinRank attacks against UOV and variants

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Generic determinantal ideals of UOV and dimension of the tangent spaces **This work**

Consider (n, m, o, q) with $n \geq 2o$ and $n - m > o$. There exists a Zariski-open set of UOV keys $\mathcal{M}_r$ such that for all $\mathcal{F} \in \mathcal{M}_r$, the ideal $I(\mathcal{O}) \cap \mathrm{Minors}_{r+1}(\mathrm{Jac}_{\mathcal{F}})$ has the expected dimension.

Supporting heuristics in the geometric analysis of MinRank attacks.

Geometry of MinRank attacks [Suzuki, Furue, Ito, Nakamura, Uchiyama 26]

- The Rectangular MinRank attacks [Beullens 22], [Furue, Ikematsu 23] are, in a way, generalizations of the singular points attack [P. 25] (which generalizes the Kipnis-Shamir attack [Kipnis, Patarin, Goubin 99]).

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  $\longrightarrow$ holds for all $\mathcal{P} \in \mathcal{U}$.

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Kipnis-Shamir and singular points [KS 98], [KPG 99], [Luyten 2023], [P. 2025]

Kipnis-Shamir attack when $n < 2o + m$.

Write $P'_i = P_i + P_i^T$. We have $P'_i\mathcal{O} \subset \mathcal{O}^\perp$ for all i. $\dim P'_i\mathcal{O} \cap P'_j\mathcal{O} = 2o + m - n > 0$ and elements of $\mathcal{O}$ can be eigenvectors of $P_j'^{-1}P'_i$.

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Geometric interpretation [Luyten 2023][P. 2025]

It holds that $\mathrm{Jac}_{\boldsymbol{x}}(\mathcal{P}) = 2(\boldsymbol{x}^T P'_1, \ldots, \boldsymbol{x}^T P'_m)^T$, thus a point $\boldsymbol{x} \in V$ is singular if and only if $\exists \boldsymbol{\lambda} \neq 0, (\lambda_1, \ldots, \lambda_m)\mathrm{Jac}_{\boldsymbol{x}}(\mathcal{P}) = 0 \in \mathbb{F}_q^n$ if and only if $\left(\sum_k \lambda_k P'_k\right)\boldsymbol{x} = 0$.

On the other hand, $\boldsymbol{x} \in \mathcal{O}$ is an eigenvector of $P_j'^{-1}P'_i$ if and only if $(P'_i - \lambda P'_j)\boldsymbol{x} = 0$.
$\rightarrow$**Kipnis-Shamir returns singular points of V.**

Experimental results.

Method

Compute Gröbner bases of ideals defined by UOV keys sampled uniformly at random in finite fields of various size and characteristic, with 1000 samples in each case.

n, m, o	r	$\mathcal{D}_{16}$	$\mathcal{D}_{31}$	$\mathcal{D}_{251}$	$\mathcal{D}_{1021}$	$\mathcal{D}_{4093}$
10, 4, 4	1	93.3%	96.5%	99.8%	100%	99.9%
10, 5, 5	1	99.9%	99.9%	100%	100%	100%
	2	93.1%	97.6%	99.4%	99.7%	100%
12, 5, 5	1	99.5%	100%	100%	100%	100%
12, 7, 5	1	100%	99.9%	100%	100%	100%
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Figure 2: Proportion of UOV keys in the Zariski open sets $\mathcal{M}_{m-r}$.

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Remark on UOV public keys forming regular sequences.

Widely reproduced experiments in the literature (see e.g. [Faugère, Perret 09]).
Unsurprisingly, we are also able to reproduce these results (with 100% success, even in $\mathbb{F}_2$).
Method: sample UOV keys uniformly and compute the Hilbert series of the sequence.

Open problems and conclusion

Weak keys from failure of genericity

[Beullens 2025] attack on SNOVA exploits failures of genericity to find increasing rank defects for decreasing proportions of public keys.

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Preliminary version in Chap. 5 of my thesis: `https://theses.hal.science/tel-05605746`.
Experiments and code on GitHub soon, ask for code by email!