
Sums of squares in polynomial time

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Hilbert's sum-of-squares problem

A homogeneous (form) $p \in \mathbb{R}[x]_{2d}$ is **nonnegative** if

$$p(x) \geq 0 \quad \text{for every } x \in \mathbb{R}^n$$

It is a **sum of squares** if

$$p = g_1^2 + \cdots + g_m^2, \quad g_i \in \mathbb{R}[x]_d$$

Every SOS form is nonnegative. **Hilbert (1888)** derived when the converse holds

$$\Sigma_{n,2d} = \mathcal{P}_{n,2d} \iff n = 2 \quad \text{or} \quad d = 1 \quad \text{or} \quad (n, d) = (3, 2)$$

Outside these three cases, SOS is a proper inner approximation of nonnegativity

History

Motzkin (1967)

$x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \geq 0$ but not SOS

Blekherman (2004)

There are many more nonnegative polynomials than sums of squares

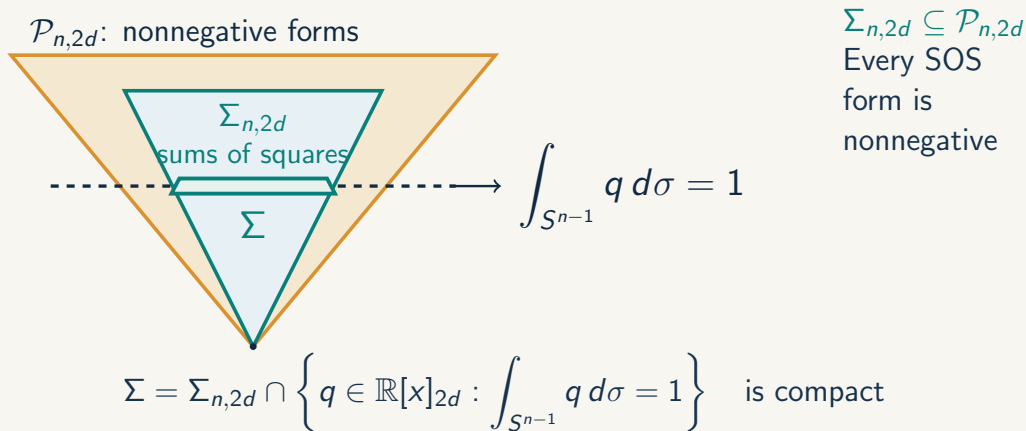
Scheiderer (2016)

There are rational SOS polynomials not having a rational SOS decomposition

Magron–Safey El Din–Schweighofer (2017)

One can compute rational univariate SOS decompositions in polynomial time

Cones and the normalized SOS slice



σ is the normalized Haar probability measure on S^{n-1}

Gram matrices for SOS forms

Work with forms of degree $2d$ in n variables

$$N_d = \binom{n+d-1}{d}$$

$$[x]_d = (x^\alpha)_{|\alpha|=d}$$

$$[x]_d^T X [x]_d \in \mathbb{R}[x]_{2d}$$

The matrix X gives Gram coordinates. The polynomial $[x]_d^T X [x]_d$ gives coefficient coordinates. The proof controls this change of coordinates.

$$q \in \Sigma_{n,2d} \Leftrightarrow q = [x]_d^T X [x]_d, X \succeq 0$$

With

$$\omega(x) = x_1^2 + \cdots + x_n^2$$

we have $\omega = 1$ on S^{n-1} . Hence $\omega^d \in \Sigma$

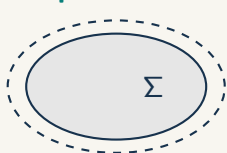
Two main complexity results

Weak membership for the slice

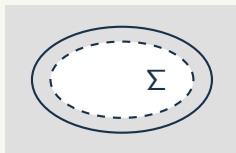
The compact set Σ has a polynomial-time weak membership algorithm in the Turing bit model

Input p with $\int_{S^{n-1}} p d\sigma = 1$, $\epsilon > 0$

Output



p is ϵ -close to Σ



p is not ϵ -deep in Σ

Interior SOS projection

The normalized SOS projection problem has a polynomial-time algorithm in the Turing bit model

Input p with $\int_{S^{n-1}} p d\sigma = 1$, $\epsilon > 0$

Output $X \succ 0$ with

$$\int_{S^{n-1}} [x]_d^T X [x]_d d\sigma = 1$$

$$\|p - [x]_d^T X [x]_d\| \leq \text{dist}(p, \Sigma) + \epsilon$$

From weak optimization to membership

Grötschel–Lovász–Schrijver (1981)

Polynomial-time **weak optimization** on Σ + controlled inner/outer radii of Σ
 $\Rightarrow$ polynomial-time **weak membership**

$$\max_{q \in \Sigma} \langle p, q \rangle,$$

where

$$\langle p, q \rangle = \sum_{\alpha} p_{\alpha} q_{\alpha}$$

A polynomial-time engine for SDP

de Klerk–Vallentin (2016)

A rational semidefinite program can be solved to accuracy ϵ in Turing-polynomial time, provided it is well rounded

What “well rounded” means

- ▶ SDP data of polynomial bit size
- ▶ $B(a_0, r) \subseteq \text{feasible set} \subseteq B(a_0, R)$
- ▶ a_0, r, R of polynomial bit size

The running time is polynomial in the input bit size, $\log(R/r)$, $\log(1/\epsilon)$

We will verify these hypotheses for the SOS slice & the projection SDP

Exact rational Haar moments

For $|\alpha| = |\beta| = d$, set $\gamma = \alpha + \beta$. Define

$$M_{\alpha,\beta} = \int_{S^{n-1}} x^\gamma d\sigma(x).$$

Here σ is the rotationally invariant probability measure on S^{n-1} .

$$M_{\alpha,\beta} = \begin{cases} 0 & \text{some } \gamma_j \text{ is odd} \\ \frac{\prod_{j=1}^n (\gamma_j - 1)!!}{n(n+2) \cdots (n+2d-2)} & \text{every } \gamma_j \text{ is even} \end{cases}$$

No numerical integration is needed. No irrational constants appear.

Each moment has polynomial bit size

Assume every γ_j is even. Then the formula has only small integer factors.

Numerator

exactly d odd factors. Each is at most $2d - 1$

Denominator

exactly d factors. Each is at most $n + 2d - 2$

$$\text{bits}(M_{\alpha,\beta}) = O(d \log(n + d))$$

Lowest terms can only shorten the encoding. Thus the full $N_d \times N_d$ matrix has size

$$O(N_d^2 d \log(n + d)) = \text{poly}(N_d)$$

for $n \geq 2$ and $d \geq 1$

The moment matrix is the L^2 metric

For $r(x) = \sum_{|\alpha|=d} c_\alpha x^\alpha$

$$c^T M c = \int_{S^{n-1}} \left(\sum_{|\alpha|=d} c_\alpha x^\alpha \right)^2 d\sigma(x) = \|r\|_{L^2}^2$$

$$\frac{c^T M c}{c^T c} = \frac{\|r\|_{L^2}^2}{\|r\|^2}$$

Rayleigh–Ritz converts norm bounds into eigenvalue bounds for M

Exact moments plus norm equivalence $\Rightarrow$ polynomial bit control

Three norms and efficient equivalence

For $p = \sum_{|\alpha|=d} p_\alpha x^\alpha$

$$\|p\|^2 = \sum p_\alpha^2 \quad \|p\|_a^2 = \frac{1}{d!} \sum \alpha! p_\alpha^2$$

$$\|p\|_{L^2}^2 = \int_{S^{n-1}} p(x)^2 d\sigma(x)$$

Two norms are **efficiently equivalent** if rational constants $k, K > 0$ are computable with polynomial bit size and

$$k\|p\|' \leq \|p\| \leq K\|p\|'$$

holds for every form p

Coefficient norm versus apolar norm

Both norms are diagonal in the monomial basis. Let

$$m_d = \min\{\alpha! : |\alpha| = d\}$$

Then

$$\frac{m_d}{d!} \|p\|^2 \leq \|p\|_a^2 \leq \|p\|^2$$

- ▶ The weights $\alpha!/d!$ are rational
- ▶ They are exactly computable for all N_d monomials
- ▶ Their bit lengths are $O(d \log d)$

L^2 norm versus apolar norm

Both inner products are $O(n)$ -invariant. Decompose degree- d forms as

$$\mathbb{R}[x]_d = \text{Harm}_d \oplus \omega \text{Harm}_{d-2} \oplus \omega^2 \text{Harm}_{d-4} \oplus \cdots, \quad \omega = x_1^2 + \cdots + x_n^2$$

On each harmonic summand there is a single scale factor:

$$(h, g)_{L^2} = A_k (h, g)_a \quad A_k > 0$$

The scale is propagated through the decomposition by $(\omega h, \omega h)_a = (h, \Delta \omega)_a$

$$\Delta(\omega^{r+1} h) = (r+1)(4r+4\ell+2n) \omega^r h \quad h \in \text{Harm}_\ell$$

The factors A_k are explicit rational numbers, and their encoding lengths are polynomially controlled

Inner and outer radii for the SOS slice

Blekherman's Loewner–John analysis gives two radii around the center ω^d :

$$\boxed{B_{L^2}(\omega^d, r_L) \subseteq \Sigma} \quad \text{and} \quad \boxed{\Sigma \subseteq B_{L^2}(\omega^d, R_L)}$$

$$r_L = \frac{d! \Gamma\left(\frac{2d+n}{2}\right)}{\Gamma\left(\frac{4d+n}{2}\right) \sqrt{N_d - 1}} \quad R_L = \sqrt{N_d - 1}.$$

Inner radius r_L : a ball contained in Σ . It gives interior room

Outer radius R_L : a ball containing Σ . It bounds the search region

Efficient norm equivalence transfers both radii to coefficient coordinates. Then $\log(R/r)$ stays polynomial in N_d

Weak optimization through eigenvalues

Weak optimization over Σ reduces to normalized squares. For $P_{\alpha,\beta} = p_{\alpha+\beta}$,

$$\begin{aligned}\max_{q \in \Sigma} \langle p, q \rangle &= \max_{c^T M c = 1} c^T P c \\ &= \max\{\langle P, X \rangle : X \succeq 0, \langle M, X \rangle = 1\}\end{aligned}$$

Useful inequalities for radii

$$\begin{aligned}\lambda_{\min}(M) \operatorname{tr}(X) &\leq \langle M, X \rangle \leq \operatorname{tr}(X) \lambda_{\max}(M) \\ \|X\|_F &\leq \operatorname{tr}(X) \leq \sqrt{n} \|X\|_F \\ \frac{1}{\sqrt{n} \lambda_{\max}(M)} &\leq \|X\|_F \leq \frac{1}{\lambda_{\min}(M)}\end{aligned}$$

A simple center

Use a diagonal matrix indexed by $|\alpha| = d$:

$$X_0 = \text{Diag}\left(\binom{d}{\alpha}\right) \quad \binom{d}{\alpha} = \frac{d!}{\alpha!}$$

Then

$$[x]_d^T X_0 [x]_d = (x_1^2 + \cdots + x_n^2)^d = \omega^d$$

$$X_0 \succ 0$$

$$\int_{S^{n-1}} [x]_d^T X_0 [x]_d d\sigma = \langle M, X_0 \rangle = 1$$

The entries are positive integers with polynomial bit size

Weak optimization yields weak membership

The SDP is well rounded, so the de Klerk–Vallentin interior-point method gives polynomial-time **weak optimization**

Blekherman's Loewner–John analysis gives two radii around the center ω^d

$$\boxed{B_{L^2}(\omega^d, r_L) \subseteq \Sigma} \quad \text{and} \quad \boxed{\Sigma \subseteq B_{L^2}(\omega^d, R_L)}$$

Grötschel–Lovász–Schrijver (1981)

Polynomial-time **weak optimization** on Σ + controlled inner/outer radii of Σ
 $\Rightarrow$ polynomial-time **weak membership**

This yields weak membership for the slice Σ

Projection as a rational SDP

The normalized projection is

$$\begin{array}{ll} \min_{X, t} & t \\ \text{s.t.} & \|p - [x]_d^T X [x]_d\| \leq t \\ & \langle M, X \rangle = 1 \quad X \succeq 0 \end{array}$$

coefficient vector u of the error

$$\begin{bmatrix} t/ & u \\ u^T & t \end{bmatrix} \succeq 0$$

Useful inequalities for radii

$$\lambda_{\min}(M) \operatorname{tr}(X) \leq \langle M, X \rangle \leq \operatorname{tr}(X) \lambda_{\max}(M)$$

$$\|X\|_F \leq \operatorname{tr}(X) \leq \sqrt{n} \|X\|_F$$

$$\frac{1}{\sqrt{n} \lambda_{\max}(M)} \leq \|X\|_F \leq \frac{1}{\lambda_{\min}(M)}$$

These inner and outer bounds make the projection SDP well rounded

Thanks for listening!

More details on [arXiv:2606.25118](#)