Positivity Proofs for Linear Recurrences through Cones

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Examples

[Straub-Zudilin 2015]

$$s_n = \sum_{k=0}^{n} (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k} \geq 0, n \in \mathbb{N}$$

Combinatorics

$$v_n = \binom{2n}{n}^2 - \frac{16^n}{4n} > 0,\ n > 1$$

Gillis-Reznick-Zeilberger family of sequences

$$u_n^{(k)} = \sum_{j=0}^{n} (-1)^j \frac{(kn - (k-1)j)!k!^j}{(n-j)!^k j!} \geq 0, n \in \mathbb{N}$$

$$k \geq 4$$

Turán's inequality

$$u_{n-1}(x) = P_n(x)^2 - P_{n-1}(x) P_{n+1}(x) > 0,\ x \in (-1,1)$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$$

Solutions of **Linear Recurrences**

Positivity Problem

Input: $\begin{cases} p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n,\ p_i \in \mathbb{Q}[n] \\ u_0, u_1, \ldots, u_{d-1} \in \mathbb{Q} \end{cases}$

Order d

Output: **True** if $\forall n \in \mathbb{N}, u_n > 0$

Recurrence to vector form

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n$$

Let $U_n = (u_n, u_{n+1}, \ldots, u_{n+d-1})^t$, then

$$U_{n+1} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{p_0(n)}{p_d(n)} & \frac{p_1(n)}{p_d(n)} & \frac{p_2(n)}{p_d(n)} & \cdots & \frac{p_{d-1}(n)}{p_d(n)} \end{pmatrix}}_{A(n)} U_n$$

Assumption: $A := \lim_{n\to\infty} A(n) \in \mathrm{GL}_d(\mathbb{Q})$

Eigenvalues of the recurrence: Eigenvalues of A

$\lambda_1, \ldots, \lambda_\nu \in \mathbb{C}$ are the **dominant eigenvalues** of A if :

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_\nu| > |\lambda_{\nu+1}| \geq |\lambda_{\nu+2}| \cdots$$

Decidability

C-finite	P-finite
$u_{n+d} = c_{d-1}u_{n+d-1} + \cdots + c_0 u_n,\ c_i \in \mathbb{Q}$	$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n, p_i \in \mathbb{Q}[n]$
$u_n = \sum q_i(n)\lambda_i^n, q_i \in \bar{\mathbb{Q}}[n]$	No general closed form

Decidability

C-finite:

- Arbitrary $d \in \mathbb{N}$ and [Ouaknine-Worrell 2014]

  $\lambda_1 > |\lambda_2| \geq \cdots$ + Generic initial conditions

- $d \leq 5$, arbitrary λ_i
- $d = 6$: Open problems in diophantine approximation

P-finite:

- $d = 2$ and cases of $d = 3$ with [Kauers-Pillwein 2010]

  $\lambda_1 > |\lambda_2| \geq \cdots$ + Generic initial conditions + λ_1 simple

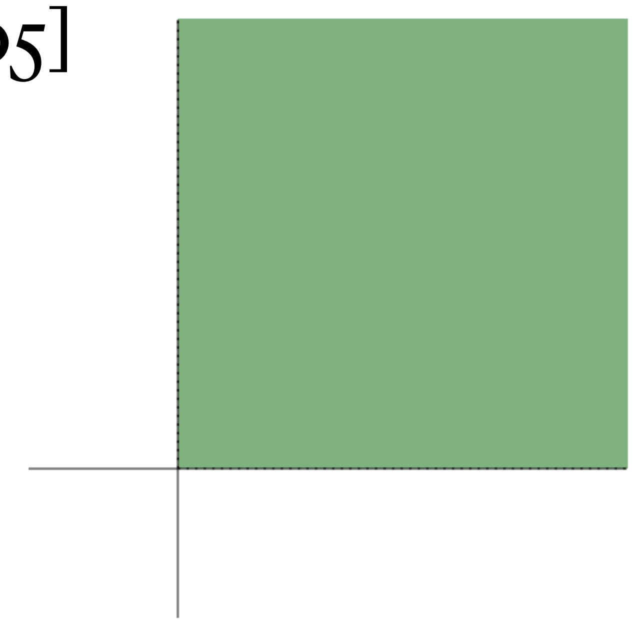
Arbitrary $d \in \mathbb{N}$?

Several dominant eigenvalues?

Proofs by induction

$$p_{d}(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n$$

[Gerhold-Kauers 2005]

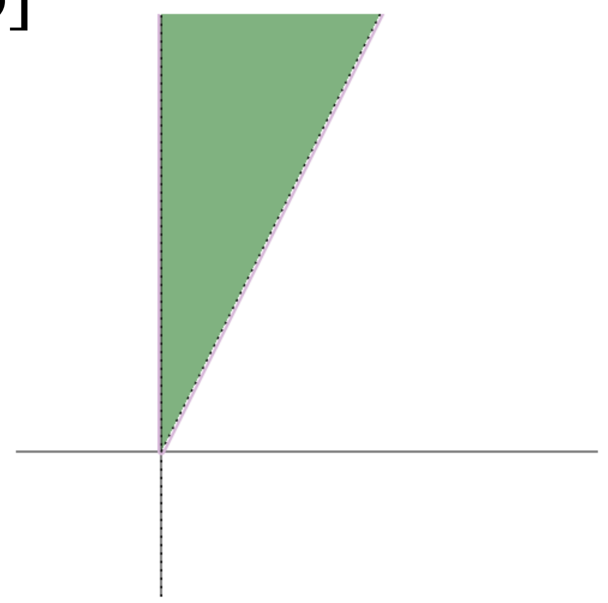


By induction: Use **quantifier elimination** to find **iteratively** $m \in \mathbb{N}$ s.t

$$\forall n \geq 0, \forall u_n \geq 0, \ldots, \forall u_{n+m} \geq 0 \implies u_{n+m+1} > 0$$

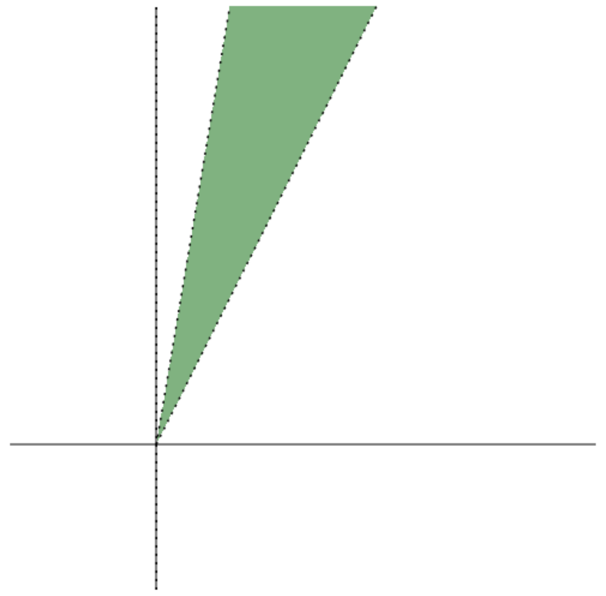
⚠ **No guarantee that m exists**

[Kauers-Pillwein 2010]



Variant: Change the induction hypothesis to $\exists \beta > 0,\ u_{n+1} > \beta\ u_n > 0$

↳ Termination (under assumptions) for $d = 2$ and cases of $d = 3$



Idea: Find "suitable" cones

Cone-based approach

$$A = \lim_{n\to\infty} A(n)$$

Input

$U_{n+1} = A(n)U_n$

U_0

Find $K \in \mathbb{R}^d_{>0}$,

$AK \subset K^\circ$

Find N

$$\begin{cases} A(n)K \subset K,\ n \geq N \\ U_N \in K \end{cases}$$

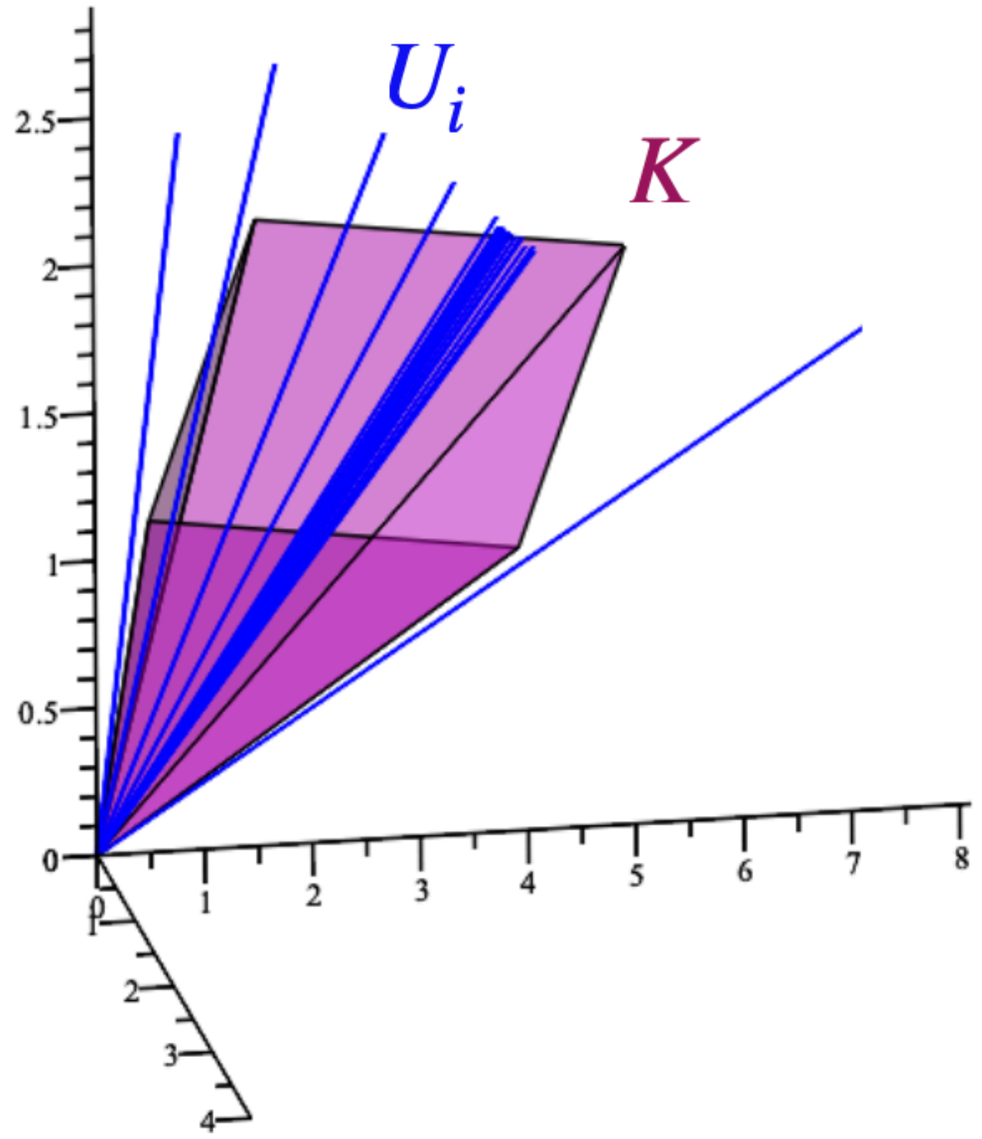
Check if

$U_0 > 0, \ldots, U_{N-1} > 0$

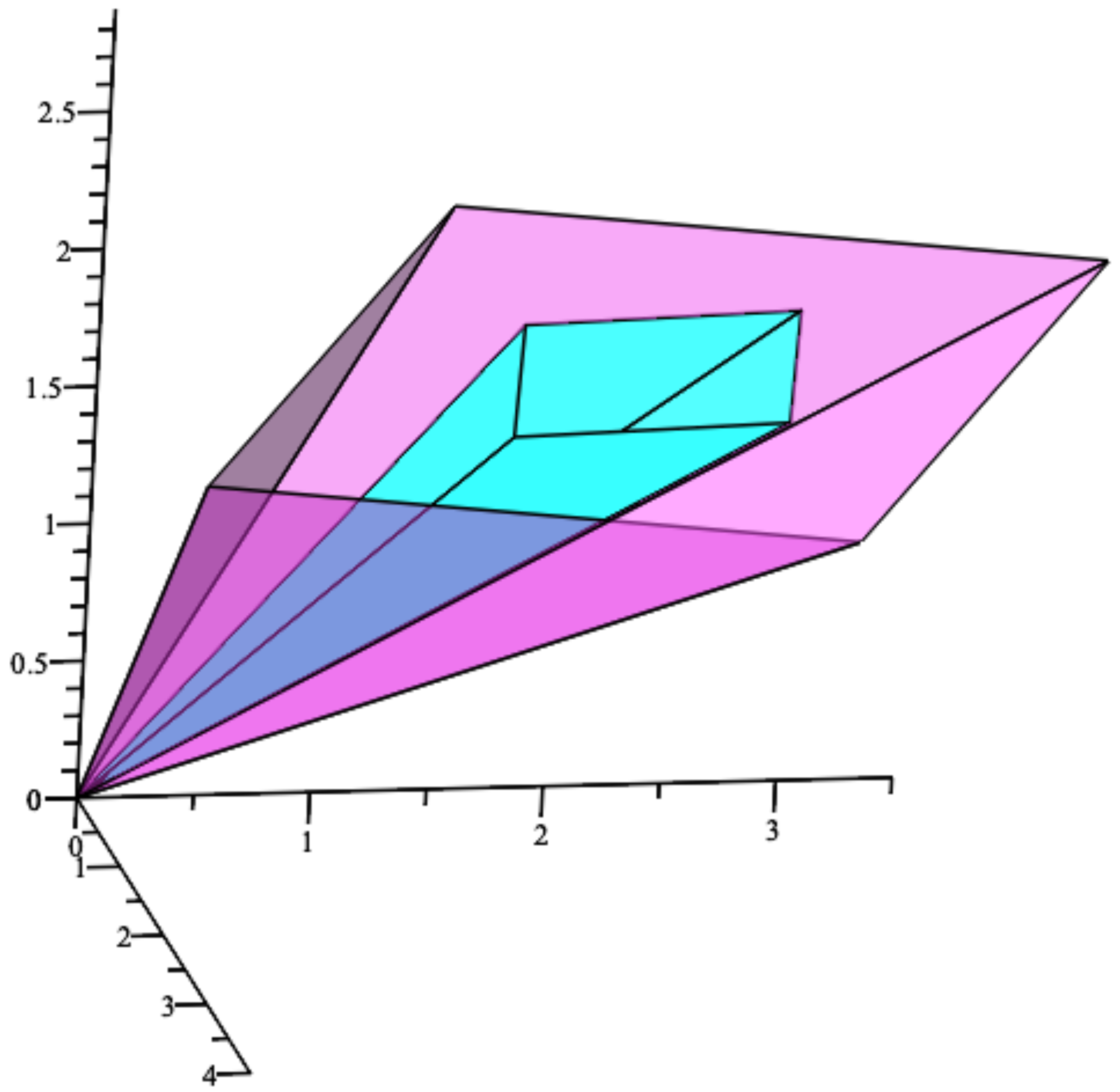
$U_n \in K, n \geq N$

Output

Positive or not



Contracted cones



$AK \backslash \{0\} \subset K^\circ$

A contracts K

$K \neq \mathbb{R}^d, K^\circ \neq \emptyset$

Theorem

There exists a proper cone K contracted by A if and only if:

. $\lambda_1 > |\lambda_2| \geq \cdots$

. λ_1 simple

[Vandergraft 68]

Explicit construction based on the Jordan form of the matrix A

K is **not unique**

Example: Cone Construction

$$(2n-1)u_{n+3} = (9n-7)u_n - (10n+9)u_{n+1} + (7n+8)u_{n+2}$$

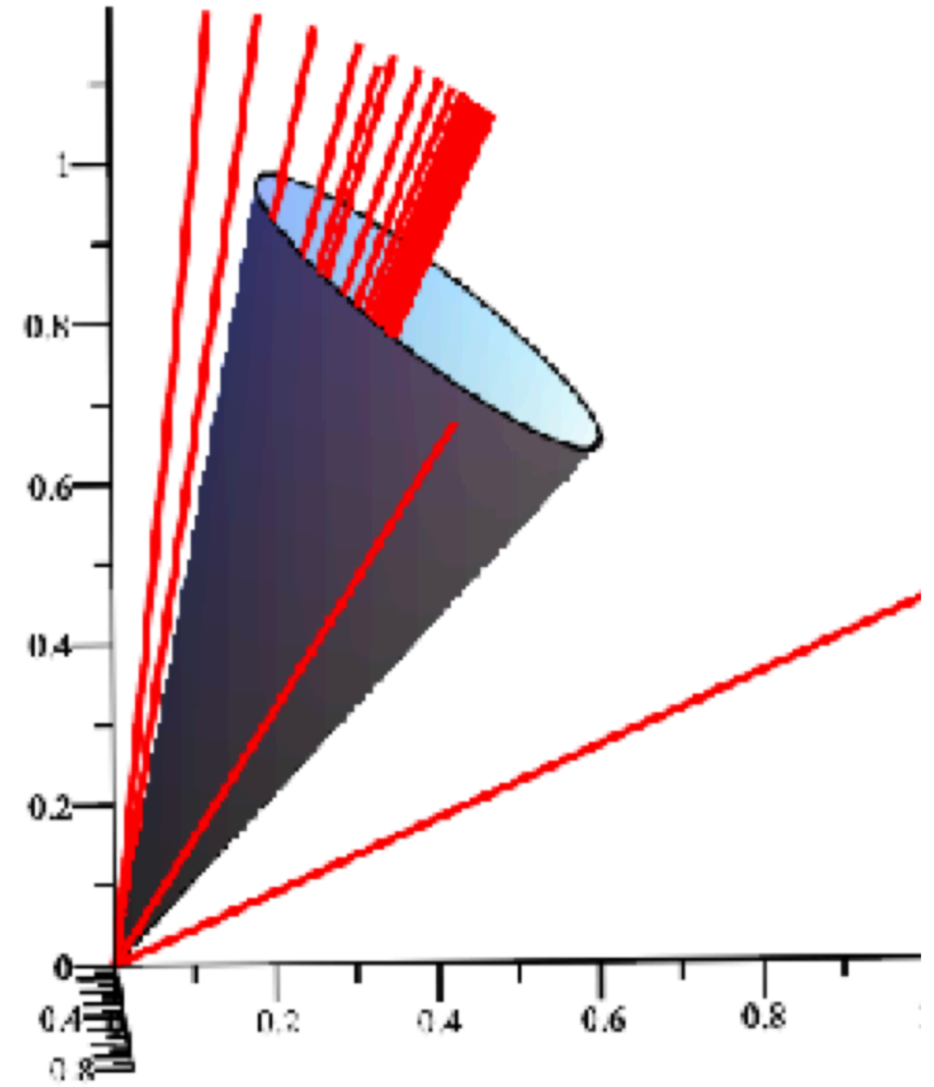
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{9}{2} & -5 & \frac{7}{2} \end{pmatrix} \qquad \begin{array}{l} \lambda_1 \sim 2.15 > |\lambda_2| \\ \lambda_2 \sim 0.68 + 1.28i \end{array}$$

Construction:

Eigenvectors: $V_1, V_2, \overline{V_2}$

$$K = \{aV_1 + bV_2 + \bar{b}\overline{V_2},\ |b| \leq a\}$$

$$A\,.(aV_1 + bV_2 + \bar{b}\overline{V_2}) = \lambda_1 a\ V_1 + \lambda_2 b\ V_2 + \overline{\lambda_2 b_2}\ \overline{V2}$$



Termination: One simple dominant eigenvalue

$$U_{n+1} = A(n)U_n, \; A = \lim_{n\to\infty} A(n) \in \mathbb{Q}^{d\times d}$$

Theorem [I.-Salvy 2024]

Positivity is decidable for $d \in \mathbb{N}$ with $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \dots$ + Generic initial conditions + λ_1 simple

Vandergraft Construction

Find $K \in \mathbb{R}^d_{>0}$ s.t

$AK \subset K^\circ$

By approximating the eigenvalues

Convergence of $A(n) \to A$

Compute n_0 s.t

$A(n)K \subset K, n \geq n_0$

Solving polynomial inequalities in $\mathbb{Q}[n]$

Generic initial conditions

[Friedland 2006]

Find $N \geq n_0$ s.t

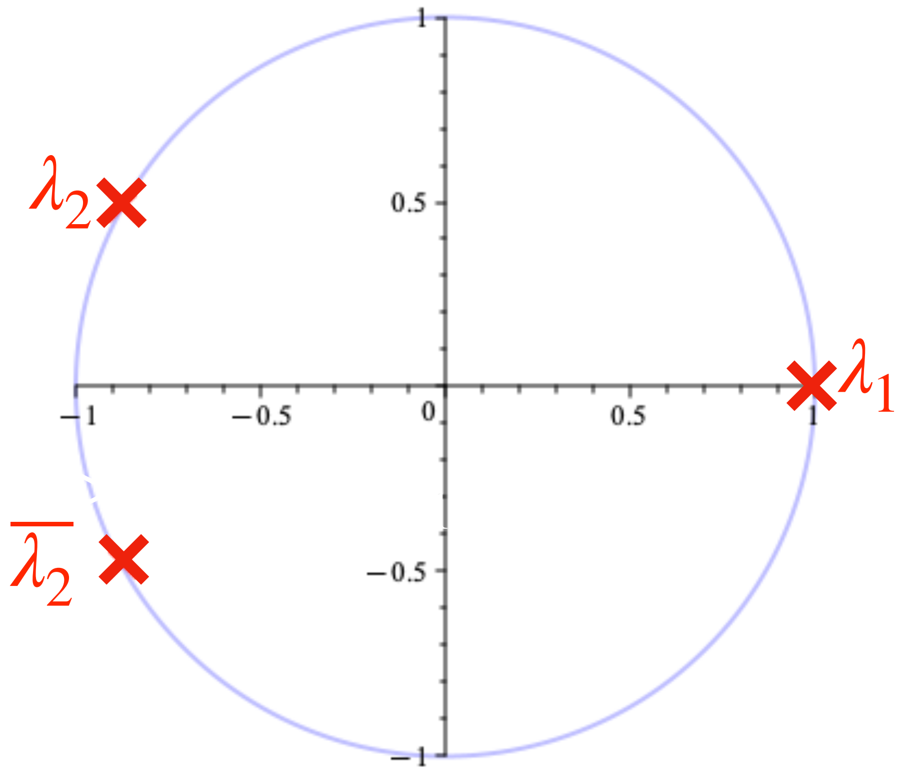
$U_N \in K$

Computations in $\mathbb{Q}$

$\lambda_1 = |\lambda_2| = \dots = |\lambda_v|$?

Example with no contracted cone

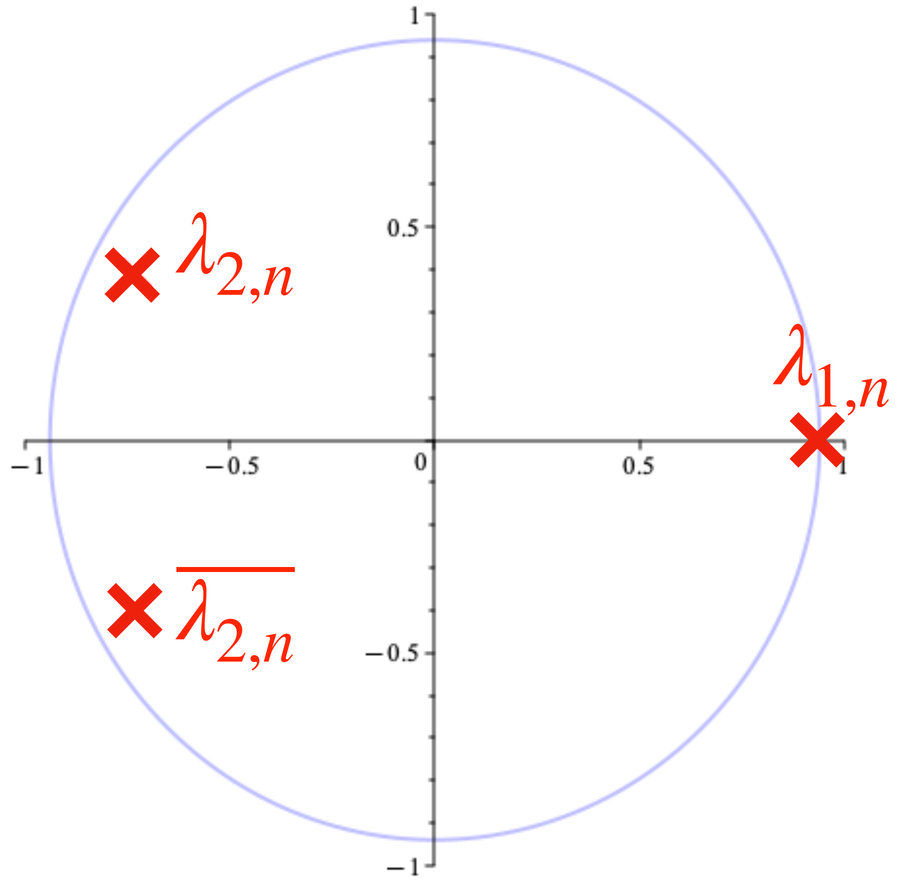
$$u_{n-1} = P_n\left(\frac{1}{4}\right)^2 - P_{n-1}\left(\frac{1}{4}\right)P_{n+1}\left(\frac{1}{4}\right),\ n \geq 1$$



~~$\lambda_1 > |\lambda_2|$~~ $\implies$ No contracted cone by A

Idea: Find $(K_n)_n \in \mathbb{R}^d_{>0} \,|\, A(n)K_n \subset K_{n+1}$

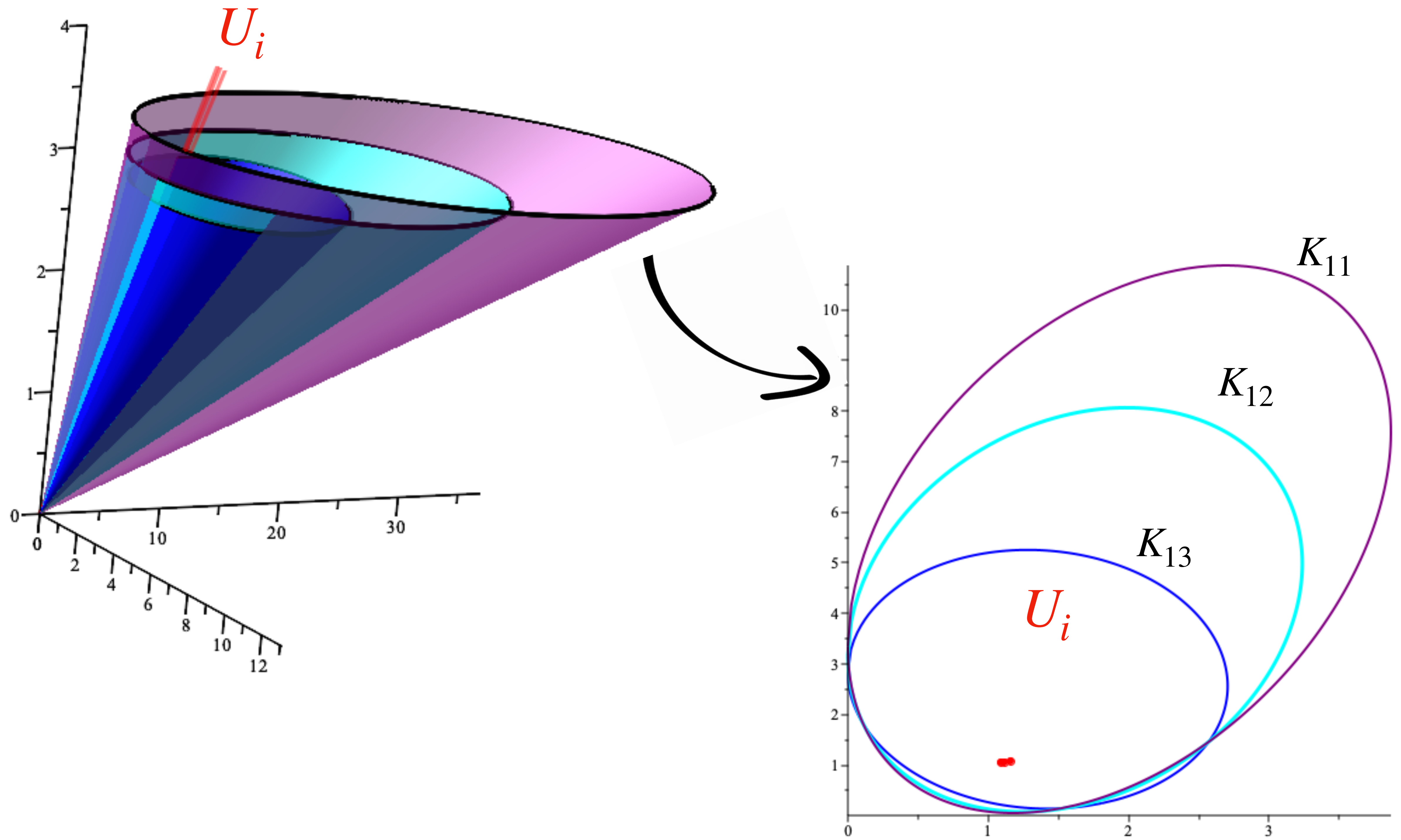
$$U_n \in K_n \implies U_{n+1} \in K_{n+1} \subset \mathbb{R}^d_{>0}$$



$\lambda_{i,n}$: Eigenvalues of $A(n)$

$\lambda_{1,n} > |\lambda_{2,n}| \implies \exists\, K_n$ contracted by $A(n)$

Cone-based approach: Extension



By induction: $\forall n \geq 11, U_n \in K_n \subset \mathbb{R}^d_{>0}$

Several Simple Dominant Eigenvalues

$$U_{n+1} = A(n)U_n,\ A = \lim_{n\to\infty} A(n) \in \mathbb{Q}^{d\times d}$$

Theorem [I. 2025]

There exists $(K_n)_n$ such that $A(n)K_n \subset K_{n+1} \subset \mathbb{R}^d_{>0}$ if

. $\lambda_{1,n} > |\lambda_{2,n}| \geq |\lambda_{3,n}| \geq \ldots \geq |\lambda_{k,n}|$

. $\lambda_1 = |\lambda_2| = \ldots = |\lambda_v| > |\lambda_j|$

. $\lambda_{1,n}, \ldots, \lambda_{v,n}$ simple

. $\max_{i=1,\ldots,k} |\lambda_{i,n} - \lambda_{i,n+1}| = o\left(\lambda_{1,n} - |\lambda_{2,n}|\right),\ n \to \infty$.

$$\lambda_i = \lim_{n\to\infty} \lambda_{i,n}$$
$$\lambda_i \neq \lambda_j$$

Conditions on the recurrence

Corollary

Positivity is decidable for recurrences of this class with arbitrary $d \in \mathbb{N}$, if additionally $\lim_{n\to\infty} \frac{U_n}{\|U_n\|} = V_1$ with $AV_1 = \lambda_1 V_1$.

Condition on the initial conditions

Examples

Recurrence	Order d	Dominant Eigenvalues	N
$s_n = \sum_{k=0}^{n} (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k}$	2	One simple	1
Gillis, Reznick, Zeilberger inequality	4 to 24	One simple	< 4 for $d \neq 5$ 546 for $d = 5$
Turán's inequality $x = 1/4$	3	Three simple	11
$v_n = \binom{2n}{n}^2 - \frac{16^n}{4n}$	2	One **double**	6
${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x^2) \geq (1+x)\ {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ $x \in (0,1)$	2	One **double**	4

Timings

[Gillis, Reznick, Zeilberger 83]

$$u_n^{(k)} = \sum_{j=0}^{n} (-1)^j \frac{(kn-(k-1)j)!\,k!^j}{(n-j)!^k\,j!} \geq 0$$

Order k

k	max deg	max $\log_{10}$ coeffs	N	$t(s)$
4	6	10	3	0.1
5	10	20	546	0.2
6	15	32	3	0.2
7	21	52	3	0.4
8	28	72	4	0.5
9	36	100	4	2.4
10	45	129	4	1.7
11	55	175	3	2.8
12	66	202	4	5.3
13	78	270	1	7.3
14	91	310	1	16.5
15	105	366	1	21.1
16	120	433	1	39.8
17	136	531	1	53.9
18	153	569	1	78.6
19	171	699	1	120.4
20	190	739	1	209.1
21	210	850	1	446.5
22	231	960	1	416.9
23	253	1115	1	731.6
24	276	1140	1	1827.2

Conclusion

This cone-based approach gives positivity proofs for a large class of sequences

Ongoing work: Extension to sequences with parameters

$$\forall x \in (-1,1),\ \ P_n(x)^2 - P_{n-1}(x)\,P_{n+1}(x) > 0, \quad n \geq 1.$$

Thank you!

$$ {}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2) - (1+x)\ {}_2F_1(\frac{1}{2},\frac{1}{2};1;x) = \sum_{n\geq 0} a_n x^n \qquad x \in (0,1) $$

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} $$

Cone-based approach: Extension

Input
$U_{n+1} = A(n)U_n$
U_0

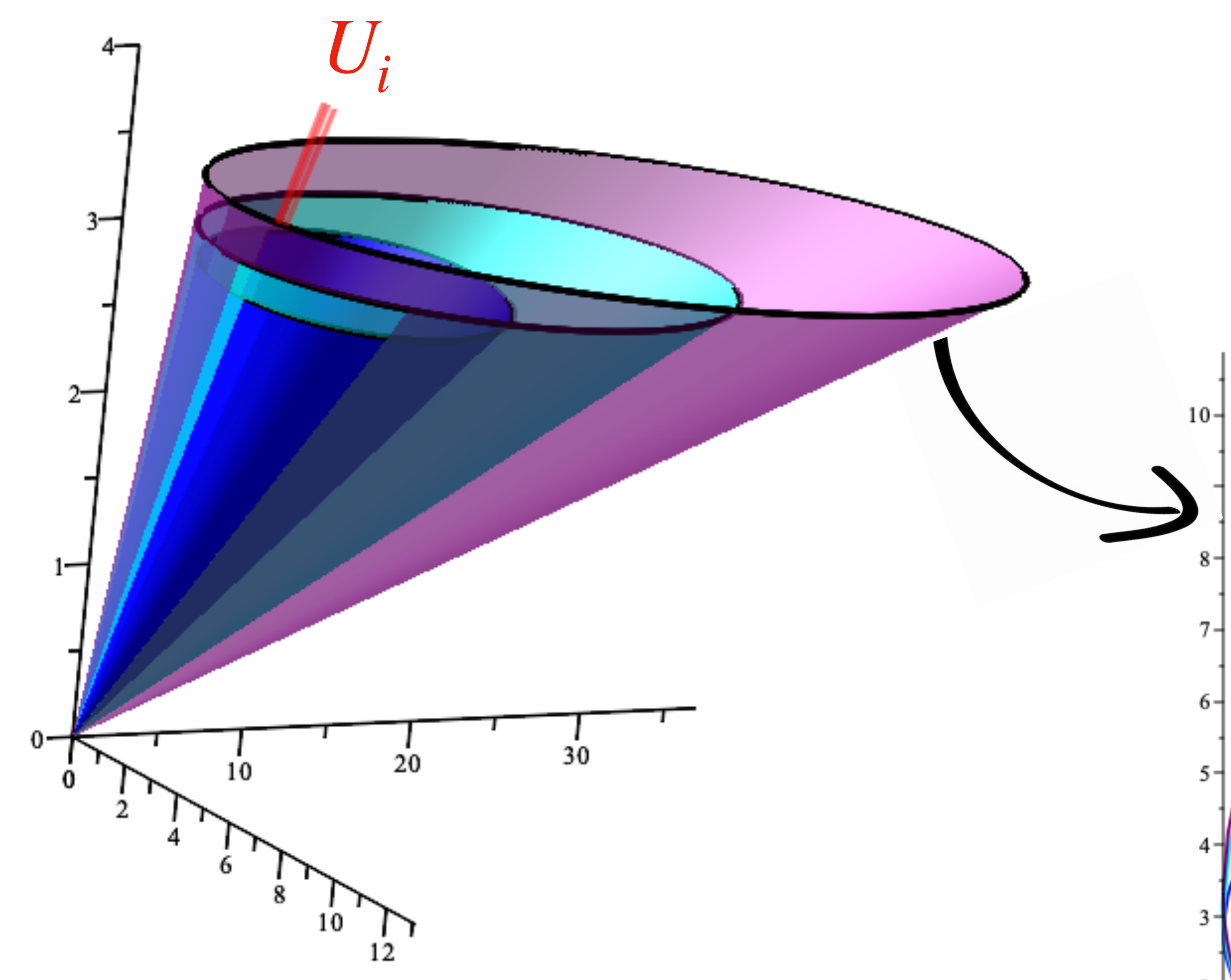
Find $K_n \in \mathbb{R}^d_{>0}$,
$A_n K_n \subset K_n^\circ$

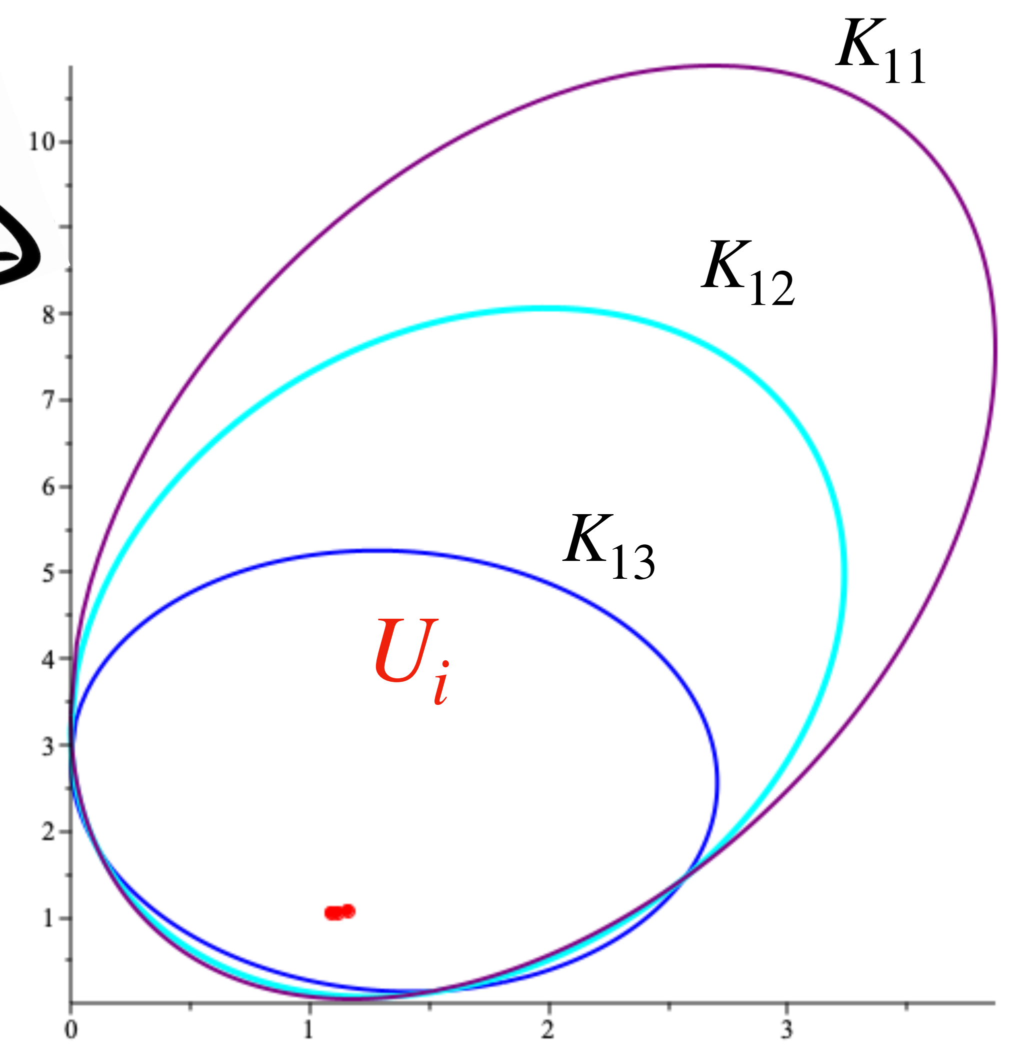
Find N
$\begin{cases} A(n)K_n \subset K_{n+1},\ n \geq N \\ U_N \in K_N \end{cases}$

Check if
$U_0 > 0, \ldots, U_{N-1} > 0$

$U_n \in K_n, n \geq N$

Output
Positif or not





By Induction: $\forall n \geq 11, U_n \in K_n \subset \mathbb{R}^d_{>0}$