A unified approach for degree bound estimates of linear differential operators

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June 19th 2025

D-finite functions

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$f(x) \in k[[x]]$ solution of

$$a_r(x)f^{(r)}(x) + \cdots + a_1(x)f'(x) + a_0(x)f(x) = 0, \quad a_i(x) \in k[x]$$

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r = order of L

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Ex (LCLM): Given L_1, L_2 s.t. $L_1(f) = 0$, $L_2(g) = 0 \rightarrow$ Find L s.t. $L(f + g) = 0$.

LCLM of differential operators

$L_1 = (x^2+1)\,\partial^2 - (x+2)\,\partial - 3$ and $L_2 = x^2\,\partial^2 - (x+3)\,\partial - 2$

Compute $L = L_1 \oplus L_2$

a minimal order operator s.t. $L(\alpha) = L(\alpha_1 + \alpha_2) = 0$ for all α_1, α_2 s.t. $L_1(\alpha_1) = L_2(\alpha_2) = 0$

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$\alpha = \alpha_1 + \alpha_2$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \eta_0 \end{bmatrix} \overset{?}{=} 0$$

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$$\alpha'' = \frac{3}{x^2+1}\alpha_1 + \frac{x+2}{x^2+1}\alpha_1' + \frac{2}{x^2}\alpha_2 + \frac{x+3}{x^2}\alpha_2'$$

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Solve the linear system and get $L = \eta_4 \cdot \partial^4 + \eta_3 \cdot \partial^3 + \eta_2 \cdot \partial^2 + \eta_1 \cdot \partial + \eta_0 \in k[x]\langle \partial \rangle$

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One specific instance of a class of algorithms

A unified algorithmic scheme

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Focus on several problems:

- Closure properties (LCLM, Symmetric product) [Stanley 1980] [van der Hoeven 2016] [Bostan, Chyzak, Li, Salvy 2012]
- Computation of a differential equation satisfied by an algebraic function [Bostan, Chyzak, Lecerf, Salvy, Schost 2007]
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Known bounds on $\deg \eta$?

- General arguments (col. by col.) $\to$ weak bounds
- *Ad hoc* arguments for special T $\to$ tight bounds

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Theorem (G. 2025)

Let $T = XM^{-1}Y$ *and* $\delta = \deg\det M$. *(+ technical assumptions in ISSAC paper)*
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Unified approach for bounds:

- Exhibit a *small* realisation $T = XM^{-1}Y$ (*i.e.* with minimal $\delta = \deg\det M$)
- Retrieve or improve the best known bound

Bounds in specific problems

Our unified approach catches the bounds!

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	Previous Bound	Our Bound	Matrix of the Problem
LCLM $\oplus$	$ds^2r + o(ds^2r)$ [BCLS 12]	$ds^2r + o(ds^2r)$	$T = \mathrm{Diag}(C_1, C_2)$
SymProd $\otimes$	$O(dr^{2s})$ [Kauers 14]	$O(dr^{2s-1})$	$T = C_1 \otimes I_{r_2} + I_{r_1} \otimes C_2$
AlgeqtoDiffeq	$4d_y^2d_x + o(d_y^2d_x)$ [BCLSS 07]	$2d_y^2d_x + o(d_y^2d_x)$	$T : a \bmod P \mapsto -\partial_y(a)P_x/P_y \bmod P$
Hermite	$2d_y^2d_x + o(d_y^2d_x)$ [BCCL 10]	$2d_y^2d_x + o(d_y^2d_x)$	$T : a \bmod Q \mapsto -\mathrm{herm}(Q_x a/Q^2)$

Denominators of minors of *pseudo*-Krylov matrices

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Key Proposition

For $s_1 \leq \cdots \leq s_r$ and $K = \begin{bmatrix} \theta^{s_1} a & \cdots & \theta^{s_r} a \end{bmatrix}$.
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Technical tool:
Determinantal denominators [Coppel 74]

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Warm-up without differentiation: the classical Krylov case

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Induction

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No such matrix factorisation in the differential case

Remark: Gives tight estimates on the size of the minimal polynomial of T

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$K = \begin{bmatrix} \theta^{s_1} u_1 & \theta^{s_2} u_2 & \cdots & \theta^{s_r} u_r \end{bmatrix}$ $\quad \theta = \partial + T \quad T = XM^{-1}Y \quad \Delta = \det M \quad u_i \in k[x]^n$

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By multilinearity of det **and induction**: Minor of $K_1 = \frac{\cdots}{\Delta^{s_r - 1}}$

$\implies$ Minor of $K = \frac{\cdots}{\Delta^{s_r}}$ (by Cauchy Binet + multilinearity)

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- Non minimal operators (ex: CLM instead of LCLM)

Thank you for your attention!