Emulation of the FMA and ADD3 in rounding-to-nearest floating-point arithmetic

Stef Graillat   Jean-Michel Muller

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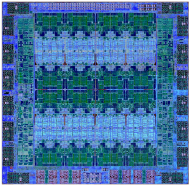
The FMA instruction... $ab + c$ with one rounding only

- correctly rounded evaluation of $ab + c$, where a, b, and c are FP numbers;
- introduced in the IBM Power1 processor (1990), and then in the Intel/HP Itanium.



- very useful:
  - software implementation of correctly rounded $\div$ and $\sqrt{}$;
    (FPN q close enough to $a/b \rightarrow a - bq$ is a FPN)
  - faster, and (in general) more accurate evaluation of dot products and polynomials;
  - accurate implementation of transcendental functions:
    $a_0 + x\rho$, where $\rho = a_1 + x(a_2 + x(a_3 + \cdots))$ and $|x|$ small.
- specified by the IEEE-754 Std on FP arithmetic since 2008
  $\rightarrow$ implemented in most computing environments;
- notable exceptions: Java Virtual Machine, WebAssembly, many microcontrollers units used for instance in automotive applications.

ADD3... $a + b + c$ with one rounding only

- correctly rounded evaluation of $a + b + c$;
- not specified by IEEE-754 $\rightarrow$ not provided by computing environments;
- and yet, would greatly help
  - final rounding step in correctly-rounded elementary functions (Lauter, 2017);
  - implementation of double-word and triple-word arithmetics (high precision numbers represented by unevaluated sums of 2 or 3 FP numbers);
- a fast hardware ADD3 would be a nice replacement for 2Sum: the error of the addition $s = \mathrm{RN}(a + b)$ is $\mathrm{RN}(a + b - s)$ computed with one ADD3. (N/A here: we are going to use 2Sum to emulate ADD3!).

Aim of this work

- software emulation of the FMA, ADD3, and the error of these operations;

  (errors: interesting for building compensated algorithms)
- high-level algorithms: FP operations/comparisons only (no use of the internal binary representation of the FP numbers);
- binary, precision-p, rounded-to-nearest FP arithmetic, with unbounded exponent range

  $\rightarrow$ our results apply to “real life” FP arithmetic provided underflow/overflow do not occur;

Notation

- set $\mathbb{F}$ of the binary, precision-p FP numbers:

$$x = M_x \cdot 2^{e_x - p + 1},$$

where $M_x, e_x \in \mathbb{Z}$, and either $M_x = 0$, or $2^{p-1} \leq |M_x| \leq 2^p - 1$;

- M_x is the integral significand of x;
- $\mathbb{F}^* = \mathbb{F} \backslash \{0\}$;
- RN: round-to-nearest, ties-to-even rounding function

  `x = y + z` $\rightarrow x = \mathrm{RN}(y + z)$

- midpoints the numbers where the value of RN changes. Exactly halfway between two consecutive FP numbers;

Notation

- unit round-off: $u = 2^{-p}$. Bounds the relative error due to rounding;
- $\mathrm{ulp}(t)$ (for $t \in \mathbb{R}$) defined as

$$\begin{cases} 0 & \text{if } t = 0, \\ 2^{\lfloor \log_2 |t| \rfloor - p + 1} & \text{otherwise;} \end{cases}$$

  - If $t \notin \mathbb{F}$, $\mathrm{ulp}(t)$ is the distance between the two consecutive FP numbers that surround t;
  - if $x \in \mathbb{F}$, x is an integer multiple of $\mathrm{ulp}(x)$;
- x is a double-word (DW) number if it is an unevaluated sum $x = x_h + x_\ell$ of two FP numbers s.t. $x_h = \mathrm{RN}(x)$.

Notation

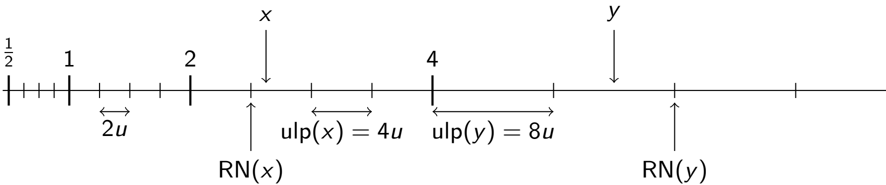


Figure 1: The floating-point numbers between $1/2$ and 8 in the toy system $p = 3$ (*i.e.*, $u = 1/8$).

Adding 3 numbers is a difficult problem!

RN-addition algorithm: only uses operations of the form

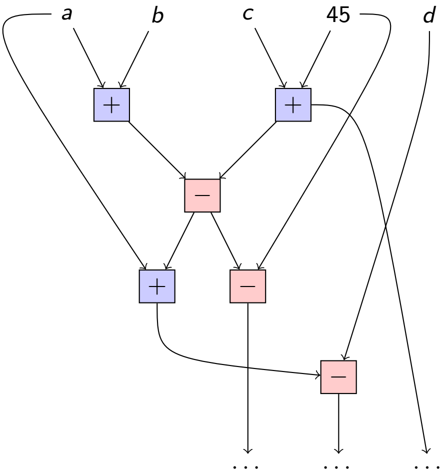
$$z \leftarrow \mathrm{RN}(\pm x \pm y)$$

(no comparisons, no tests).

Kornerup, Lefèvre, Louvet, M. (2013):

> *In binary FP arithmetic with unbounded exponent range, no RN-addition algorithm returns* $\mathrm{RN}(a+b+c)$ *for all* $a, b, c \in \mathbb{F}$.

RN-addition algorithm: DAG whose vertices are FP $+$ or $-$

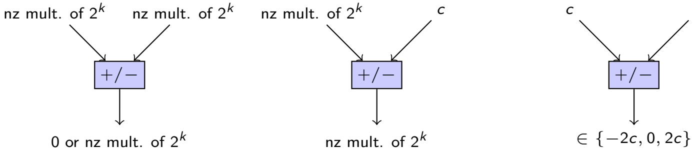


Computing RN($a + b + c$) with a RN-add algorithm?

- DAG of depth r;
- possible constants are nonzero (nz) multiples of 2^k for some $k \in \mathbb{Z}$;
- $a = 2^{k+p}$ and $b = 2^k$ ($\rightarrow a + b$ is a midpoint);
- $|c| \leq 2^{k-p-r}$

$\rightarrow$ RN($a + b + c$) is the FP number immediately below or above $a + b$, depending on the sign of c.

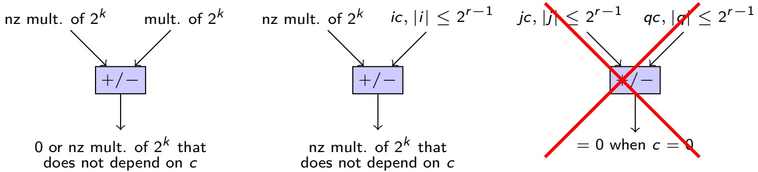
- Operations at depth 1 in the DAG:



- after depth-1 operations, available operands are multiples of 2^k that do not depend on c and elements of $\{-2c, 0, c, 2c\}$.

Computing RN($a + b + c$) with a RN-add algorithm?

- Induction: after depth-m operations, available operands are multiples of 2^k that do not depend on c and elements of $\{-2^m c, \ldots, 0, \ldots, 2^m c\}$.
- Last operation, that outputs the final result:
  - its two inputs are not both elements of $\{-2^{r-1}c, \ldots, 0, \ldots, 2^{r-1}c\}$, because when $c = 0$ it must output RN($a + b$);
  - $\rightarrow$ at least one input is a nonzero multiple of 2^k that does not depend on c;
  - $\rightarrow$ the output is a multiple of 2^k that does not depend on c.



- the sign of c cannot change the result.

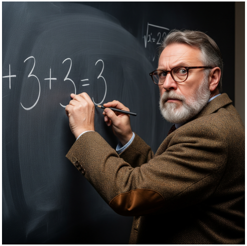
But in real life, the exponents are bounded...

Same reasoning: Assuming extremal exponents e_{min} and e_{max}, an RN-addition algorithm of depth r cannot always return $\text{RN}(a+b+c)$ as soon as

$$r \leq e_{\text{max}} - e_{\text{min}} - 2p.$$

Binary64/double precision: an RN-addition algorithm that always returns $\text{RN}(a+b+c)$ (if such an algorithm exists!) has depth ≥ 1939.

$\rightarrow$ Adding 3 numbers is a difficult problem! We cannot avoid tests/comparisons and/or use of various rounding functions.



Classical results

Theorem 1 (Sterbenz Theorem)
Let $a, b \in \mathbb{F}$. If

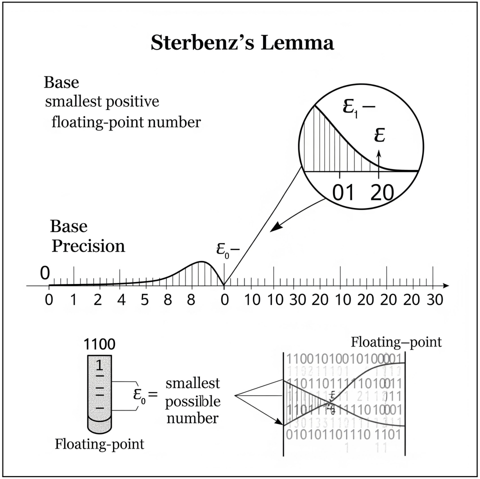
$$\frac{a}{2} \leq b \leq 2a,$$

then $a - b \in \mathbb{F}$.

Implies that the subtraction $a - b$ is performed exactly in FP arithmetic.

I have asked Gemini to illustrate Sterbenz' theorem...

Computer arithmetic, AI and Surrealism



Classical results

Theorem 2 (The error of an addition is exactly representable)
Let a and b be two FP numbers. Let

$$s = \mathrm{RN}(a+b) \text{ and } r = (a+b) - s.$$

If no overflow when computing s, then r is a FP number.

Beware: does not always work with rounding functions $\neq$ RN.

Proof: assume $|b| \leq |a|$ without l.o.g., remark that a, b, s and therefore r are multiple of $\mathrm{ulp}(b)$, and note that $|r| \leq |b|$ (because s is closer to $a+b$ than any other FP number, including a).

Obtaining r: Fast2Sum and 2Sum

$x_h \leftarrow \mathrm{RN}(a+b)$
$z \leftarrow \mathrm{RN}(x_h - a)$
$x_\ell \leftarrow \mathrm{RN}(b - z)$
return (x_h, x_ℓ)

Alg. 1: Fast2Sum(a, b). Returns $(x_h, x_\ell) \in \mathbb{F}^2$ such that x_h is the FP number nearest $a+b$, and, if $|a| \geq |b|$ or $a = 0$, $x_\ell = (a+b) - x_h$.

$x_h \leftarrow \mathrm{RN}(a+b)$
$a' \leftarrow \mathrm{RN}(x_h - b)$
$b' \leftarrow \mathrm{RN}(x_h - a')$
$\delta_a \leftarrow \mathrm{RN}(a - a')$
$\delta_b \leftarrow \mathrm{RN}(b - b')$
$x_\ell \leftarrow \mathrm{RN}(\delta_a + \delta_b)$
return (x_h, x_ℓ)

Alg. 2: 2Sum(a, b). Returns $(x_h, x_\ell) \in \mathbb{F}^2$ such that x_h is the FP number nearest $a+b$, and $x_\ell = (a+b) - x_h$.

Proof of Fast2Sum (assuming $|b| \leq |a|$)

$x_h \leftarrow \mathrm{RN}(a+b)$
$z \leftarrow \mathrm{RN}(x_h - a)$
$x_\ell \leftarrow \mathrm{RN}(b - z)$
return (x_h, x_ℓ)

- if a and b have same sign, then $|a| \leq |a+b| \leq |2a|$ hence ($2a$ is a FP number, rounding is increasing) $|a| \leq |x_h| \leq |2a| \rightarrow$ (Sterbenz) $z = x_h - a$. Since $r = (a+b) - x_h$ is a FPN and $b - z = r$, we find $x_\ell = \mathrm{RN}(b-z) = r$.
- if a and b have opposite signs then
  1. either $|b| \geq \frac{1}{2}|a|$, which implies (Sterbenz) $a+b$ is a FPN, thus $x_h = a+b$, $z = b$ and $x_\ell = 0$;
  2. or $|b| < \frac{1}{2}|a|$, which implies $|a+b| > \frac{1}{2}|a|$, hence $x_h \geq \frac{1}{2}|a|$ ($\frac{1}{2}a$ is a FPN, rounding is increasing), thus (Sterbenz) $z = \mathrm{RN}(x_h - a) = x_h - a = b - r$. Since $r = (a+b) - x_h$ is a FPN and $b - z = r$,we get $x_\ell = \mathrm{RN}(b-z) = r$.

Classical results 3: The Dekker-Veltkamp multiplication

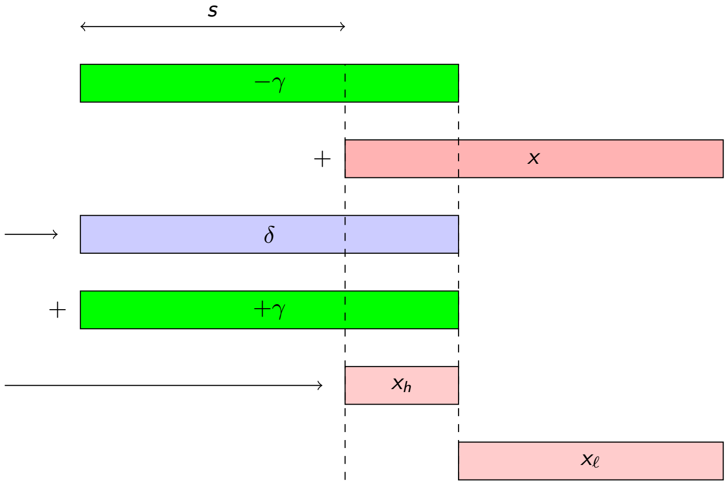
Require: $K = 2^s + 1$
Require: $2 \le s \le p - 2$
  $\gamma \leftarrow \mathrm{RN}(K \cdot x)$
  $\delta \leftarrow \mathrm{RN}(x - \gamma)$
  $x_h \leftarrow \mathrm{RN}(\gamma + \delta)$
  $x_\ell \leftarrow \mathrm{RN}(x - x_h)$
  return (x_h, x_ℓ)

Alg. 3: Split(x, s). x_h fits in $p - s$ bits, x_ℓ fits in $s - 1$ bits, $x_h + x_\ell = x$.

Require: $s = \lceil p/2 \rceil$
  $(a_h, a_\ell) \leftarrow \mathrm{Split}(a, s)$
  $(b_h, b_\ell) \leftarrow \mathrm{Split}(b, s)$
  $\pi_h \leftarrow \mathrm{RN}(a \cdot b)$
  $t_1 \leftarrow \mathrm{RN}(-\pi_h + \mathrm{RN}(a_h \cdot b_h))$
  $t_2 \leftarrow \mathrm{RN}(t_1 + \mathrm{RN}(a_h \cdot b_\ell))$
  $t_3 \leftarrow \mathrm{RN}(t_2 + \mathrm{RN}(a_\ell \cdot b_h))$
  $\pi_\ell \leftarrow \mathrm{RN}(t_3 + \mathrm{RN}(a_\ell \cdot b_\ell))$
  return (π_h, π_ℓ)

Alg. 4: DekkerProd(a, b). $\pi_h = \mathrm{RN}(ab)$ and $\pi_h + \pi_\ell = ab$.

Rough explanation of the splitting algorithm



(A bit less) classical result using Round-to-Odd

$$\mathrm{RO}(t) = \begin{cases} t \text{ if } t \in \mathbb{F} \\ \text{the FPN with an odd integral significand nearest } t \text{ otherwise.} \end{cases}$$

- not specified by IEEE-754
- not implemented in current processors of commercial significance
- known emulation uses internal binary representation (*not doable with "high level" algorithms?*)
- many nice properties, among them:

Theorem 3 (Boldo & Melquiond, 2008)

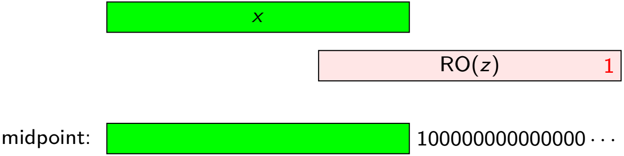
Let $x \in \mathbb{F}$ and $z \in \mathbb{R}$. If $p \geq 4$ and $6 \cdot |z| \leq x$ then

$$\mathrm{RN}(x + \mathrm{RO}(z)) = \mathrm{RN}(x + z).$$

Let $x \in \mathbb{F}$ and $z \in \mathbb{R}$. If $p \geq 4$ and $6 \cdot |z| \leq x$ then

$$\mathrm{RN}(x + \mathrm{RO}(z)) = \mathrm{RN}(x + z).$$

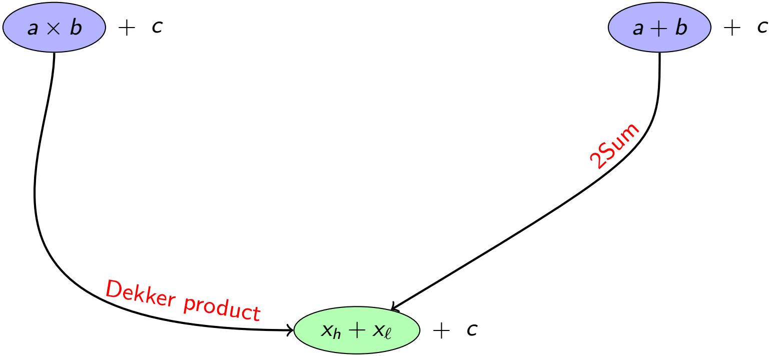
- trivial if z is a FP number, since $\mathrm{RO}(z) = z$;
- if z is not a FP number then $\mathrm{RO}(z)$ has an odd integral significand $\rightarrow x + \mathrm{RO}(z)$ is not a midpoint



- the distance between $x + z$ and $x + \mathrm{RO}(z)$ is $< \mathrm{ulp}(z)$;
- as $x + \mathrm{RO}(z)$ and the midpoints around x are multiple of $\mathrm{ulp}(z)$, the midpoint nearest $x + \mathrm{RO}(z)$ is at a distance $\geq \mathrm{ulp}(z)$ from $x + \mathrm{RO}(z)$;

$\rightarrow$ no midpoint between $x + z$ and $x + \mathrm{RO}(z)$.

FMA and ADD3 $\rightarrow$ DW number+ FP number



$\rightarrow$ we focus on: compute $x_h + x_\ell + c$, where $|x_\ell| \leq \frac{1}{2}\mathrm{ulp}(x_h)$.

Easy if we have Round-to-Odd...

$$\mathrm{RN}(x_h + x_\ell + c)$$

- define $(s_h, s_\ell) = 2Sum(x_h, c)$, so that $x_h + x_\ell + c = s_h + s_\ell + x_\ell$;
- Theorem 3 implies

$$\mathrm{RN}(s_h + s_\ell + x_\ell) = \mathrm{RN}(s_h + \mathrm{RO}(s_\ell + x_\ell)).$$

- Boldo and Melquiond give a solution for emulating $\mathrm{RO}(s_\ell + x_\ell)$.

```
struct DW
{double h, l;};

double OddRoundSum (double x, double y){
INTDOUBLE myvh;
struct DW v;
v = TwoSum(x,y);
myvh.real = v.h;
if (v.l != 0.0){
if (!(myvh.integer & 1)){
if ((v.h > 0.0) ^ (v.l < 0.0)){myvh.integer++;}
  else {myvh.integer--;}}
  }
return myvh.real;
}
```

Determining if $x \in \mathbb{F}^* = \pm 2^k$ or $\pm 3 \cdot 2^k$, $k \in \mathbb{Z}$

Theorem 4

If $p \geq 4$, the number $x \in \mathbb{F}^*$ is of the form $\pm 2^k$ or $\pm 3 \cdot 2^k$, with $k \in \mathbb{Z}$, if and only if

$$\mathrm{RN}\left[\mathrm{RN}\left(\left(2^{p-2}+1\right)\cdot x\right) - 2^{p-2}x\right] = x. \tag{1}$$

- Nonzero FPN x: odd integer N_x times power of 2;
- if $N_x \leq 3$ then $(2^{p-2}+1)N_x \leq 2^p - 1$ (as soon as $p \geq 4$), hence $(2^{p-2}+1)x$ is aFPN;
- if $N_x \geq 5$ then $(2^{p-2}+1)N_x$ is an odd integer $> 2^p \rightarrow (2^{p-2}+1)x$ is not a FPN;
- Sterbenz theorem $\rightarrow$ subtraction is exact.

Condition $p \geq 4$ is necessary: a counterexample with $p = 3$ is $x = 6$.

Determining if $x \in \mathbb{F}^* = \pm 2^k$ or $\pm 3 \cdot 2^k$, $k \in \mathbb{Z}$

Require: $P = 2^{p-2} + 1$
Require: $Q = 2^{p-2}$
  $L \leftarrow \mathrm{RN}(P \cdot x)$
  $R \leftarrow \mathrm{RN}(Q \cdot x)$
  $\Delta \leftarrow \mathrm{RN}(L - R)$
  return $(\Delta \neq x)$

Alg. 5: IsNot1or3TimesPowerOf2(x). Returns true iff $|x| \neq 0$ or 2^k or $3 \cdot 2^k$.

Computation of $\mathrm{RN}(x_h + x_\ell + c)$

$(s_h, s_\ell) \leftarrow \mathrm{2Sum}(x_h, c)$
$(v_h, v_\ell) \leftarrow \mathrm{2Sum}(x_\ell, s_\ell)$
if $\mathrm{IsNot1or3TimesPowerOf2}(v_h)$ **or** $v_\ell = 0$ **then**
    $z \leftarrow \mathrm{RN}(s_h + v_h)$
else
    if v_ℓ and v_h have the same sign **then**
        $z \leftarrow \mathrm{RN}\left(s_h + \mathrm{RN}\left(1.125 v_h\right)\right)$
    else
        $z \leftarrow \mathrm{RN}\left(s_h + \mathrm{RN}\left(0.875 v_h\right)\right)$
    end if
end if
return z

Alg. 6: CR-DWPlusFP(x_h, x_ℓ, c). Computes $\mathrm{RN}(x_h + x_\ell + c)$.

Remarks:

- $x_h + x_\ell + c = s_h + v_h + v_\ell$;
- The constants $1.125 = 9/8$ and $0.875 = 7/8$ that appear in Alg. 6 are exactly representable as soon as $p \geq 4$;

Analysis of Algorithm CR-DWPlusFP

Theorem 5

If $p \geq 5$, the number z returned by Algorithm CR-DWPlusFP (Algorithm 6) satisfies

$$z = \mathrm{RN}(x_h + x_\ell + c).$$

We just give a sketch of the proof.

Define $\Sigma = \mathrm{RN}(x_h + x_\ell + c)$.
First, 2Sum $\rightarrow$ variables s_h, v_h and v_ℓ in Algorithm 6 satisfy

$$s_h + v_h + v_\ell = x_h + x_\ell + c, \text{ so that } \Sigma = \mathrm{RN}(s_h + v_h + v_\ell),$$
$$|v_\ell| \leq \frac{1}{2}\mathrm{ulp}(v_h).$$

$\rightarrow$ discussion on $s_h + v_h + v_\ell$.

Analysis of Algorithm CR-DWPlusFP

$(s_h, s_\ell) \leftarrow \mathrm{2Sum}(x_h, c)$
$(v_h, v_\ell) \leftarrow \mathrm{2Sum}(x_\ell, s_\ell)$
if IsNot1or3TimesPowerOf2(v_h) **or** $v_\ell = 0$ **then**
  $z \leftarrow \mathrm{RN}(s_h + v_h)$
else
  if v_ℓ and v_h have the same sign **then**
    $z \leftarrow \mathrm{RN}\left(s_h + \mathrm{RN}\left(\frac{9}{8} v_h\right)\right)$
  else
    $z \leftarrow \mathrm{RN}\left(s_h + \mathrm{RN}\left(\frac{7}{8} v_h\right)\right)$
  end if
end if
return z

- If $p \geq 5$, when $v_h = \pm 2^k$ or $\pm 3 \cdot 2^k$, the terms $(7/8)v_h$ and $(9/8)v_h$ are FP numbers;
- case $x_h = 0$ straightforward;
- If x_h, x_ℓ, and c are multiplied by $\pm 2^k$, then s_h, s_ℓ, v_h, v_ℓ, z and Σ are multiplied by $\pm 2^k$;
- If we interchange x_h and c, same result $\rightarrow$ prove the theorem in the case $|x_h| \geq |c|$;

$\Rightarrow$ We focus on $1 \leq x_h \leq 2 - 2u$ and $|c| \leq x_h$.

Analysis of Algorithm CR-DWPlusFP

We focus on $1 \le x_h \le 2 - 2u$ and $|c| \le x_h$.

$(s_h, s_\ell) \leftarrow \mathrm{2Sum}(x_h, c)$
$(v_h, v_\ell) \leftarrow \mathrm{2Sum}(x_\ell, s_\ell)$
if IsNot1or3TimesPowerOf2(v_h) **or** $v_\ell = 0$ **then**
  $z \leftarrow \mathrm{RN}(s_h + v_h)$
else
  if v_ℓ and v_h have the same sign **then**
    $z \leftarrow \mathrm{RN}\left(s_h + \mathrm{RN}\left(\frac{9}{8} v_h\right)\right)$
  else
    $z \leftarrow \mathrm{RN}\left(s_h + \mathrm{RN}\left(\frac{7}{8} v_h\right)\right)$
  end if
end if
return z

- If $-x_h \le c \le -\frac{x_h}{2}$, Sterbenz $\Rightarrow s_\ell = 0 \Rightarrow$ straightforward;
- we focus on

$$-\frac{x_h}{2} < c \le x_h,$$

which implies

$$\frac{1}{2} \le s_h \le 4 - 4u.$$

Analysis of Algorithm CR-DWPlusFP

$(s_h, s_\ell) \leftarrow \text{2Sum}(x_h, c)$
$(v_h, v_\ell) \leftarrow \text{2Sum}(x_\ell, s_\ell)$
if IsNot1or3TimesPowerOf2(v_h) **or** $v_\ell = 0$ **then**
    $z \leftarrow \text{RN}(s_h + v_h)$
else
    if v_ℓ and v_h have same sign **then**
        $z \leftarrow \text{RN}\left(s_h + \text{RN}\left(\frac{9}{8} v_h\right)\right)$
    else
        $z \leftarrow \text{RN}\left(s_h + \text{RN}\left(\frac{7}{8} v_h\right)\right)$
    end if
end if
return z

Reminder: $\Sigma = \text{RN}(s_h + v_h + v_\ell)$, and $\frac{1}{2} \leq s_h \leq 4 - 4u$

- Case A: $\frac{1}{2} \leq s_h \leq 1 - u$, s_h multiple of u, $|s_\ell| \leq \frac{u}{2}$, $|v_h| \leq \frac{3u}{2}$, $|v_\ell| \leq u^2$;
- Case B: $1 \leq s_h \leq 2 - 2u$, s_h mult. of $2u$, $|s_\ell| \leq u$, $|v_h| \leq 2u$, $|v_\ell| \leq u^2$;
- Case C: $2 \leq s_h \leq 4 - 4u$, s_h mult. of $4u$, $|s_\ell| \leq 2u$, $|v_h| \leq 3u$, $|v_\ell| \leq 2u^2$.
- In all cases distance between $s_h + v_h$ and a midpoint multiple of $\text{ulp}(v_h)$. As $|v_\ell| \leq \frac{1}{2}\text{ulp}(v_h)$, If $s_h + v_h$ is not a midpoint, no midpoint between $s_h + v_h$ and $s_h + v_h + v_\ell \rightarrow \Sigma = \text{RN}(s_h + v_h)$.

When can $s_h + v_h$ be a midpoint?

The midpoints are the odd multiples of $\frac{u}{4}$ in $[\frac{1}{4}, \frac{1}{2})$, the odd multiples of $\frac{u}{2}$ in $[\frac{1}{2}, 1)$, and the odd multiples of u in $[1, 2)$. Therefore, if $s_h + v_h$ is a midpoint:

- In Case A: if $s_h = \frac{1}{2}$ then $v_h \in \{-\frac{3u}{4}, -\frac{u}{4}, \frac{u}{2}, \frac{3u}{2}\}$, and if $\frac{1}{2} < s_h \leq 1 - u$ then $v_h \in \{-\frac{3u}{2}, -\frac{u}{2}, \frac{u}{2}, \frac{3u}{2}\}$;
- In Case B: if $s_h = 1$ then $v_h \in \{-\frac{3u}{2}, -\frac{u}{2}, u\}$, and if $1 < s_h \leq 2 - 2u$ then $v_h \in \{-u, u\}$;
- In Case C: if $s_h = 2$ then $v_h \in \{-3u, -u, 2u\}$ and if $2 < s_h \leq 4 - 4u$ then $v_h \in \{-2u, 2u\}$.

In all cases, v_h is of the form $\pm 2^k$ or $\pm 3 \cdot 2^k$.

Analysis of Algorithm CR-DWPlusFP

$(s_h, s_\ell) \leftarrow \text{2Sum}(x_h, c)$
$(v_h, v_\ell) \leftarrow \text{2Sum}(x_\ell, s_\ell)$
if IsNot1or3TimesPowerOf2(v_h) **or** $v_\ell = 0$ **then**
  $z \leftarrow \text{RN}(s_h + v_h)$
else
  if v_ℓ and v_h have the same sign **then**
    $z \leftarrow \text{RN}\left(s_h + \text{RN}\left(\frac{9}{8} v_h\right)\right)$
  else
    $z \leftarrow \text{RN}\left(s_h + \text{RN}\left(\frac{7}{8} v_h\right)\right)$
  end if
end if
return z

- when v_h is not of the form $\pm 2^k$ or $\pm 3 \cdot 2^k$, $s_h + v_h$ is not a midpoint, so that

$$\Sigma = \text{RN}(s_h + v_h);$$

- when v_h is of the form $\pm 2^k$ or $\pm 3 \cdot 2^k$, case-by-case analysis.

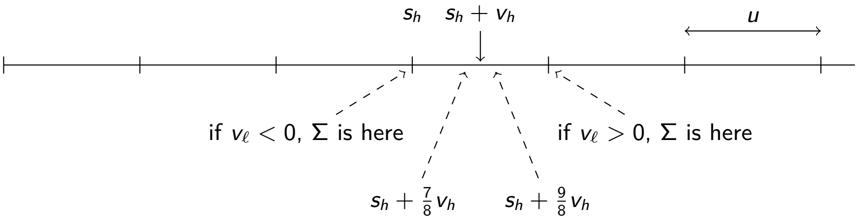


Figure 2: The subcase $\frac{1}{2} < s_h < 1 - u$ and $v_h = +\frac{u}{2}$.

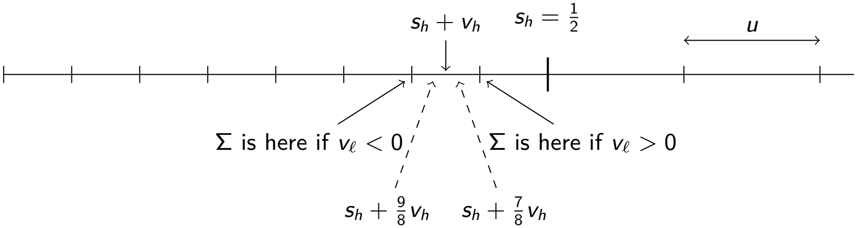


Figure 3: The subcase $s_h = \frac{1}{2}$ and $v_h = -\frac{3u}{4}$.

Emulation of ADD3 and the FMA

1: $(x_h, x_\ell) \leftarrow \text{2Sum}(a, b)$
2: **return** $\text{CR-DWPlusFP}(x_h, x_\ell, c)$

Alg. 7: Computation of $\text{RN}(a + b + c)$.

1: $(x_h, x_\ell) \leftarrow \text{DekkerProd}(a, b)$
2: **return** $\text{CR-DWPlusFP}(x_h, x_\ell, c)$

Alg. 8: Computation of $\text{FMA}(a, b, c) = \text{RN}(ab + c)$.

Theorem 6

In a binary, precision-p (with $p \geq 5$), FP arithmetic, Alg. 7 returns $\text{RN}(a + b + c)$ and Alg. 8 returns $\text{RN}(ab + c)$ for all a, b, $c \in \mathbb{F}$.

Error of these operations?

Same as previously: we reduce FMA and ADD3 to $x_h + x_\ell + c$.

$(s_h, s_\ell) \leftarrow \mathrm{2Sum}(x_h, c)$
$(v_h, v_\ell) \leftarrow \mathrm{2Sum}(x_\ell, s_\ell)$
$(w_h, w_\ell) \leftarrow \mathrm{Fast2Sum}(s_h, v_h)$
if IsNot1or3TimesPowerOf2(v_h) **or** $v_\ell = 0$ **then**
    $\delta \leftarrow w_\ell$
    $z \leftarrow w_h$
else
    if v_ℓ and v_h have the same sign **then**
        $z \leftarrow \mathrm{RN}\left(s_h + \mathrm{RN}\left(\frac{9}{8} v_h\right)\right)$
    else
        $z \leftarrow \mathrm{RN}\left(s_h + \mathrm{RN}\left(\frac{7}{8} v_h\right)\right)$
    end if
    $\alpha \leftarrow \mathrm{RN}(z - w_h)$
    $\delta \leftarrow \mathrm{RN}(w_\ell - \alpha)$
end if
return (z, δ, v_ℓ)

Alg. 9: CR-DWPlusFP-with-error. Computes $z = \mathrm{RN}(x_h + x_\ell + c)$, and δ and v_ℓ such that $z + \delta + v_\ell = x_h + x_\ell + c$.

Error of these operations?

Theorem 7

If $p \geq 5$ *and* $|x_\ell| \leq \frac{1}{2}\mathrm{ulp}(x_h)$, *the numbers* z, δ, *and* v_ℓ *returned by Algorithm 9 satisfy*

$$z = \mathrm{RN}(x_h + x_\ell + c)$$

and

$$\delta + v_\ell = x_h + x_\ell + c - z.$$

1: $(x_h, x_\ell) \leftarrow \mathrm{DekkerProd}(a, b)$
2: **return** $\mathrm{CR\text{-}DWPlusFP\text{-}with\text{-}error}(x_h, x_\ell, c)$

Alg. 10: – **FMA-with-error**(a, b, c). Computes $z = \mathrm{RN}(ab + c)$ and δ and v_ℓ such that $ab + c = z + \delta + v_\ell$.

Error of the FMA *when there is a hardware FMA*

- an algorithm was suggested by Boldo and M. in 2005;
- Alg. 10: Dekker Product then Alg. 9 $\rightarrow RN(ab + c)$ and error of that operation;
- however, on platforms with an FMA:
  - to obtain x_h and x_ℓ, no need to use Dekker product, since $x_h = \mathrm{RN}(ab)$ and $x_\ell = ab - x_h$ are obtained with a multiplication and an FMA;
  - the tests needed to compute z in Alg. 9 are no longer necessary: $z = \mathrm{RN}(ab + c)$ is obtained with an FMA;
  - ...other simplifications that would need to dive into the proof of Theorem 7.

Error of the FMA *when there is a hardware FMA*

$z_h \leftarrow \mathrm{RN}(ab + c)$
$x_h = \mathrm{RN}(ab)$
$x_\ell = \mathrm{RN}(ab - x_h)$
$(s_h, s_\ell) \leftarrow \mathrm{2Sum}(x_h, c)$
$(v_h, v_\ell) \leftarrow \mathrm{2Sum}(x_\ell, s_\ell)$
$\alpha' \leftarrow \mathrm{RN}(z_h - s_h)$
$\delta' \leftarrow \mathrm{RN}(v_h - \alpha')$
return (z, δ', v_ℓ)

Alg. 11: Computes $z = \mathrm{RN}(ab + c)$ and δ' and v_ℓ s.t. $ab + c = z + \delta' + v_\ell$.

Our ADD3 vs Boldo and Melquiond's

Table 1: Time (in seconds) to perform 5×10^9 ADD3 operations in binary64, using the Boldo-Melquiond algorithm and Algorithm 7, on different environments. Each operand: $K \times s \times F$, where F is uniform in $[0,1]$, $s = \pm 1$ (each with probability 1/2), and $K \in \{1, 2^{\pm 20}, 2^{\pm 40}, 2^{\pm 60}, 2^{\pm 80}\}$ (each with probability 1/9).

Architecture/System	compiler and options	Boldo-Melquiond	Algorithm 7
Intel Corei7	clang (v. 16.0.0)	177	153
under MacOS	clang -O3	22	19
Apple M3Pro	clang (v. 16.0.0)	142	144
under MacOS	clang -O3	7	9
AMD Opteron 6272	gcc (v. 12.2.0)	759	659
under Linux	gcc -O3	127	104
	clang -O3	168	93
Intel Xeon Gold 6444Y	gcc (v. 12.2.0)	95	84
under Linux	gcc -O3	18	20
	clang -O3 (v. 14.0.6)	18	15

Our FMA vs Boldo and Melquiond's

Table 2: Time (in seconds) to perform 5×10^9 FMA operations in binary64, using the Boldo-Melquiond (BM) algorithm, Algorithm 8, and the FMA provided by the environment . Each operand: $K \times s \times F$, where F is uniform in $[0, 1]$, $s = \pm 1$ (each with probability 1/2), and $K \in \{1, 2^{\pm 20}, 2^{\pm 40}, 2^{\pm 60}, 2^{\pm 80}\}$ (each with probability 1/9).

Architecture/System	compiler and options	BM	Alg. 8	native
Intel Corei7	clang (v. 16.0.0)	258	200	41
under MacOS	clang -O3	31	25	10
Apple M3Pro	clang (v. 16.0.0)	228	162	7
under MacOS	clang -O3	10	9	4
AMD	gcc (v. 12.2.0)	1068	856	75
Opteron 6272	gcc -O3 -lm	190	110	42
under Linux	clang -O3 -lm	181	95	43
Intel Xeon	gcc -lm (v. 12.2.0)	109	98	10
Gold 6444Y	gcc -O3 -lm	25	24	10
under Linux	clang -O3 -lm (v. 14.0.6)	25	21	10

Our error of the FMA vs Boldo and M.'s

Table 3: Time (in seconds) to compute 5×10^9 errors of FMA operations in binary64, using the Boldo-Muller algorithm and Algs 10 and 11, on different environments. Each operand: $K \times s \times F$, where F is uniform in $[0, 1]$, $s = \pm 1$ (each with probability 1/2), and $K \in \{1, 2^{\pm 20}, 2^{\pm 40}, 2^{\pm 60}, 2^{\pm 80}\}$ (each with probability 1/9).

Architecture/System	compiler and options	Boldo-M.	Alg. 10	Alg. 11
Intel Corei7	clang (v. 16.0.0)	166	280	168
under MacOS	clang -O3	30	39	33
Apple M3Pro	clang (v. 16.0.0)	151	298	143
under MacOS	clang -O3	7	13	7
AMD Opteron 6272	gcc (v. 12.2.0)	742	1252	736
under Linux	gcc -O3	134	143	140
	clang -O3	117	122	130
Intel Xeon Gold 6444Y	gcc (v. 12.2.0)	89	144	87
under Linux	gcc -O3	23	30	24
	clang -O3 (v. 14.0.6)	22	28	22

Discussion

- primary disadvantage of Algs 6, 7, 8, and 10: presence of tests;
- however:
  - we have seen that ADD3 with only rounded to nearest $+/-$ is impossible;
  - the test (*is some variable of the form* $\pm 2^k$ *or* $\pm 3 \cdot 2^k$*?*) almost always returns **false** $\rightarrow$ in practice branch prediction works very well.
- ADD3 and FMA:
  - performance slightly better or similar (ADD3), or always better (FMA) than Boldo and Melquiond's algs (because they test on parity, which is much harder to predict?)
  - high-level algorithms.
- Error of the FMA:
  - if no hardware FMA, no real choice;
  - if hardware FMA: check on your environment, with your applications;
- better error bounds for several double-word or triple-word algorithms.

ADD3 can be simplified if an FMA is available

- Because it makes it possible to directly and very easily check if $s_h + v_h$ is a midpoint.
- $\cdots$ but that's another story, that will also allow us to compute $\mathrm{RN}(ab + cd)$ in a similar fashion!